# An exact solution for a particle in a velocity-dependent force field 

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(Received 7 June 2020; accepted 8 August 2021)


#### Abstract

We revisit the classical mechanics problem of a particle moving under the influence of a force that depends on its velocity. Using the properties of the rotation matrix and associated operators, we show that it is possible to find an exact analytical solution to a number of problems where the differential equation of motion depends on the velocity. First, we apply our method to the wellknown cases of a particle under the influence of the Lorentz force and Coriolis force, providing the complete analytical solution in each case for the motion of the particle in three dimensions. We also show that the complete solution can be obtained when the centrifugal force is included, showing the applicability to cases where there is simultaneous dependence on the position and velocity. This method, which is not currently discussed in a typical course in elementary mechanics, provides an alternative approach to the traditional methods that are used to solve these types of problems. © 2021 Published under an exclusive license by American Association of Physics Teachers. https://doi.org/10.1119/10.0005992


## I. INTRODUCTION

Many seemingly unrelated physical phenomena are described by the same equations. Richard Feynman referred to this fact as the underlying unity of nature and famously observed "the same equations have the same solutions." ${ }^{1}$ Such is the case of a particle experiencing a force that depends on its velocity, for instance, in the well-known cases of the Coriolis ${ }^{2-4}$ and Lorentz ${ }^{5,6}$ forces. There is an exact analogy between these equations of motion and those that govern the motion of a harmonic oscillator, ${ }^{7,8}$ spin systems, ${ }^{9,10}$ and others. Although the approaches used to integrate these equations vary, this unity allows us to view them as the same problem and, therefore, applies the same technique to solve them.
In general, problems addressed in introductory mechanics courses deal with the motion of a particle in an external force field that is either constant or depends only on the particle position. Typical examples of this are the harmonic oscillator and a particle in a constant gravitational or electric field. However, interesting problems exist where the force depends on the velocity of the particle, and such problems are not generally addressed in great detail in textbooks. The deflection of the trajectory of falling object by the Coriolis force and the motion of a charged particle in an electromagnetic field are certainly the most common examples, but textbooks often restrict this class of problems to particular cases where analytical solutions can be found ${ }^{11-13}$ or rely on approximations to simplify the equations of motion. However, the exact solution to a number of these problems can be found using elemental properties of vectors and matrices without the need for a special symmetry axis.

In this work, we provide a straightforward method to find the exact solution to the equations of motion of a number of dynamical systems, where the force acting on the particle
depends on the velocity. Although there are other solution techniques, this one has some useful physical examples. We will show that when the equation of motion can be written in terms of linear operators acting on the velocity and position vectors, and those operators can be expressed in terms of projectors related to the rotation matrix, we can use a number of interesting properties associated with these operators to find the exact solution to the differential equations. As an example, we apply this method to the case of a charged particle in an electromagnetic field, a falling object in the presence of a Coriolis and centrifugal forces, and a particle subjected to a generalized lift force.

## II. THEORETICAL FRAMEWORK

Consider a particle moving in three dimensions, subjected to a force that depends on its instantaneous velocity, such as the Lorentz force in electrodynamics,

$$
\begin{equation*}
m \frac{\mathrm{~d} \mathbf{v}}{\mathrm{~d} t}=q(\mathbf{E}+\mathbf{v} \times \mathbf{H}) \tag{1}
\end{equation*}
$$

Throughout this manuscript, we use boldface text to denote vector quantities. This equation of motion can be thought of as a particular case of a more general family of equations of the form

$$
\begin{align*}
\dot{\mathbf{r}}(t) & =\mathbf{v}(t)  \tag{2a}\\
\dot{\mathbf{v}}(t) & =\mathbf{a}+\mathbb{B} \mathbf{v}(t) \tag{2b}
\end{align*}
$$

which have to be solved for the initial conditions $\mathbf{r}(0)=\mathbf{r}_{0}$ and $\mathbf{v}(0)=\mathbf{v}_{0}$. Here, $\mathbf{a}$ is an acceleration vector that is assumed to be constant in time while $\mathbb{B}$ is a $3 \times 3$ matrix of real, constant elements. A formal solution to these equations
of motion can be found by multiplying both sides of Eq. (2b) by the integrating factor $\mathrm{e}^{-\mathbb{B} t}$, so that it can be rewritten as $d\left[\mathrm{e}^{-\mathbb{B} t} \mathbf{v}(t)\right] / d t=\mathrm{e}^{-\mathbb{B} t} \mathbf{a}$. Here, the exponential of a matrix is defined by the Taylor series of this function, such that every term of the series is well defined, as it is an integer power of the matrix in the argument. Direct integration and a change of variables in the integral leads to

$$
\begin{equation*}
\mathbf{v}(t)=\mathrm{e}^{t \mathbb{B}} \mathbf{v}_{0}+\left(\int_{0}^{t} \mathrm{e}^{\xi \mathbb{B}} \mathrm{d} \xi\right) \mathbf{a} . \tag{3}
\end{equation*}
$$

Integrating again, we obtain

$$
\begin{equation*}
\mathbf{r}(t)=\mathbf{r}_{0}+\left(\int_{0}^{t} \mathrm{e}^{\tau \mathbb{B}} \mathrm{d} \tau\right) \mathbf{v}_{0}+\left(\int_{0}^{t} \int_{0}^{\tau} \mathrm{e}^{\xi \mathbb{B}} \mathrm{d} \xi \mathrm{~d} \tau\right) \mathbf{a} \tag{4}
\end{equation*}
$$

which represents the most general expression for the position of the particle.

These integrals can be solved easily in terms of elementary functions if we are able find a manageable expression for $\mathrm{e}^{\tau \mathbb{B}}$, but that will depend on the properties of the operator $\mathbb{B}$. Finding the exponential of a matrix usually involves matrix diagonalization. Interestingly, diagonalizing the operator is not the only way of solving these types of problems. In fact, it is instructive to realize that in this case, it is not even necessary to obtain the eigenvalues explicitly. In Ref. 14, Gantmacher uses the characteristic polynomial in combination with the Cayley-Hamilton theorem, which can be used to rewrite the exponential of the operator, leading to an expression that is easier to handle for integration.

Here, we solve these integrals using a different method from the aforementioned approaches. This method is based on the properties of $\mathbb{B}$ and its relationships with the rotation matrix, bypassing the diagonalization procedure. To solve the integrals in Eq. (4), we will consider two different approaches, based on whether $\mathbb{B}$ is invertible or not.

## A. The operator $\mathbb{B}$ is invertible

If $\mathbb{B}^{-1}$ exists, then $\int_{0}^{\tau} \mathrm{e}^{\xi \mathbb{B}} d \xi=\left(\mathbb{B}^{-1}\right) \mathrm{e}^{\tau \mathbb{B}}-\mathbb{B}^{-1}$ and we can easily integrate Eq. (4) to obtain

$$
\begin{equation*}
\mathbf{r}(t)=\mathbf{r}_{0}-t \mathbb{B}^{-1} \mathbf{a}-\mathbb{B}^{-1}\left(\mathbb{I}-\mathrm{e}^{t \mathbb{B}}\right)\left(\mathbf{v}_{0}+\mathbb{B}^{-1} \mathbf{a}\right) \tag{5}
\end{equation*}
$$

where $\mathbb{I}$ is the identity operator. This is an exact solution to the motion of a particle of mass $m$ subjected to a force $\mathbf{F}=m(\mathbf{a}+\mathbb{B} \mathbf{v})$, in the case where the operator $\mathbb{B}^{-1}$ exists. Notice that this solution is valid even if the operator $\mathbb{B}$ is not diagonalizable.

## $B$. The operator $\mathbb{B}$ is not invertible

In general, $\mathbb{B}$ has no inverse, so the evaluation of the integrals in Eq. (4) is not as straightforward as in Subsection II A. The most common approach to deal with these types of problems is to try to diagonalize $\mathbb{B}$ in order to obtain a more manageable expression for $\mathrm{e}^{\tau \mathbb{B}}$. This process ${ }^{14}$ requires the calculation of eigenvalues and eigenvectors of $\mathbb{B}=\mathbb{P} \mathbb{P}^{-1}$, where $\mathbb{P}$ is the matrix of eigenvectors and $\mathbb{A}$ is a diagonal matrix with the eigenvalues of $\mathbb{B}$ as elements. This leads to the expression $\mathrm{e}^{\tau \mathbb{B}}=\mathrm{e}^{\tau \mathbb{P} A \mathbb{P}^{-1}}=\mathbb{P}^{\tau \mathbb{A}} \mathbb{P}^{-1}$ that is easier to integrate in Eq. (4). If $\mathbb{B}$ is not diagonalizable, it can be written in terms of its Jordan normal form, $\mathbb{J}$, such that $\mathrm{e}^{\tau \mathbb{B}}=\mathrm{e}^{\tau \mathbb{P} J \mathbb{P}^{-1}}=\mathbb{P}^{\tau \mathrm{J}} \mathbb{P}^{-1}$ can still, in principle, be integrated.

Here, we will tackle this problem from a different perspective. The idea behind our approach is that, when $\mathbb{B}$ can be written in terms of certain projection operators or the rotation matrix, we can take advantage of the multiple properties that exist between these operators to simplify the term $\mathrm{e}^{\xi \mathbb{B} B}$, such that Eq. (4) can be solved by direct integration without the need of expressing these matrices in a particular basis. This approach represents an alternative method.

Let us study the properties of the spatial rotation operator $\mathbb{R}(\hat{\mathbf{n}}, \varphi)$, for which the transformation $\mathbf{r}^{\prime}=\mathbb{R}(\hat{\mathbf{n}}, \varphi) \mathbf{r}$ rotates the vector $\mathbf{r}$ counterclockwise about the $\hat{\mathbf{n}}$ axis by an angle $\varphi$. Here, $\hat{\mathbf{n}}=\left(n_{1}, n_{2}, n_{3}\right)$ is a unit vector, and the angle $\varphi$ is in the interval $[0,2 \pi)$. This transformation can be written in terms of Rodrigues' rotation formula ${ }^{13}$

$$
\begin{align*}
\mathbf{r}^{\prime} & =\mathbb{R}(\hat{\mathbf{n}}, \varphi) \mathbf{r} \\
& =(\hat{\mathbf{n}} \cdot \mathbf{r}) \hat{\mathbf{n}}+\hat{\mathbf{n}} \times(\mathbf{r} \times \hat{\mathbf{n}}) \cos \varphi+(\hat{\mathbf{n}} \times \mathbf{r}) \sin \varphi \tag{6}
\end{align*}
$$

Defining the operations

$$
\begin{align*}
& \mathbb{N}(\hat{\mathbf{n}}) \mathbf{r} \equiv(\hat{\mathbf{n}} \cdot \mathbf{r}) \hat{\mathbf{n}}  \tag{7a}\\
& \mathbb{P}(\hat{\mathbf{n}}) \mathbf{r} \equiv \hat{\mathbf{n}} \times(\mathbf{r} \times \hat{\mathbf{n}}),  \tag{7b}\\
& \mathbb{D}(\hat{\mathbf{n}}) \mathbf{r} \equiv \mathbf{r} \times \hat{\mathbf{n}} \tag{7c}
\end{align*}
$$

where $\mathbb{N}(\hat{\mathbf{n}})$ is a projector parallel to $\hat{\mathbf{n}}, \mathbb{P}(\hat{\mathbf{n}})$ is a projector perpendicular to $\hat{\mathbf{n}}$, and $\mathbb{D}(\hat{\mathbf{n}}) \mathbf{r}$ is the dual of $\hat{\mathbf{n}}$, we have

$$
\begin{equation*}
\mathbb{R}(\hat{\mathbf{n}}, \varphi)=\mathbb{N}(\hat{\mathbf{n}})+\cos \varphi \mathbb{P}(\hat{\mathbf{n}})-\sin \varphi \mathbb{D}(\hat{\mathbf{n}}) \tag{8}
\end{equation*}
$$

In Appendix A, we provide a derivation of this formula and the explicit form of the matrices. It is easy to check that these matrices commute with each other and are non-invertible with the exception of $\mathbb{R}(\hat{\mathbf{n}}, \varphi)$, whose inverse is $\mathbb{R}(\hat{\mathbf{n}},-\varphi)$. They also satisfy the following properties (see Appendix A):

$$
\begin{align*}
& \mathbb{N}^{2}=\mathbb{N}  \tag{9}\\
& \mathbb{P}^{2}=\mathbb{P}  \tag{10}\\
& \mathbb{D}^{2}=-\mathbb{P}  \tag{11}\\
& \mathbb{D}^{3}=-\mathbb{D}  \tag{12}\\
& \mathbb{P}+\mathbb{N}=\mathbb{I}  \tag{13}\\
& \mathrm{e}^{\alpha \mathbb{P}(\hat{\mathbf{n}})}=\mathbb{N}(\hat{\mathbf{n}})+\mathrm{e}^{\alpha} \mathbb{P}(\hat{\mathbf{n}}) \tag{14}
\end{align*}
$$

It is worth noting that $\mathbb{N}(\hat{\mathbf{n}}) \mathbb{P}(\hat{\mathbf{n}})=\mathbb{N}(\hat{\mathbf{n}}) \mathbb{D}(\hat{\mathbf{n}})=0$ and $\mathbb{P}(\hat{\mathbf{n}}) \mathbb{D}(\hat{\mathbf{n}})=\mathbb{D}(\hat{\mathbf{n}})$, which leads to $\mathbb{N}(\hat{\mathbf{n}}) \mathbb{R}(\hat{\mathbf{n}}, \varphi)=\mathbb{N}(\hat{\mathbf{n}})$ and $\mathbb{P}(\hat{\mathbf{n}}) \mathbb{R}(\hat{\mathbf{n}}, \varphi)=\mathbb{R}(\hat{\mathbf{n}}, \varphi)-\mathbb{N}(\hat{\mathbf{n}})$. We can use these properties to simplify the solutions presented later. Using the equations above, we can deduce that Eq. (8) can be rewritten as

$$
\begin{equation*}
\mathbb{R}(\hat{\mathbf{n}}, \varphi)=\mathbb{I}-\sin \varphi \mathbb{D}(\hat{\mathbf{n}})+(1-\cos \varphi) \mathbb{D}^{2}(\hat{\mathbf{n}}) \tag{15}
\end{equation*}
$$

which, in turn, can be used to prove that

$$
\begin{equation*}
\mathbb{R}(\hat{\mathbf{n}}, \varphi)=\mathrm{e}^{-\varphi \mathbb{D}(\hat{\mathbf{n}})} \tag{16}
\end{equation*}
$$

or in other words, that $\mathbb{D}$ is the generator of $\mathbb{R}$.
We can use these properties to solve the integrals in Eq. (4), in particular, for the cases in which the force on the
particle involves a cross product with the velocity (the operator $\mathbb{B}$ is proportional to $\mathbb{D}$ ), that is, when $\mathbb{B}$ is skewsymmetric; that is, its transpose equals its negative. In this case, $\mathbb{B}=\alpha \mathbb{D}(\hat{\mathbf{n}})$, and the integrands in Eq. (4) can be written as

$$
\begin{align*}
\mathrm{e}^{\tau \mathbb{B}} & =\mathrm{e}^{\tau \alpha \mathbb{D}(\hat{\mathbf{n}})} \\
& =\mathbb{R}(\hat{\mathbf{n}},-\tau \alpha) \\
& =\mathbb{I}+\sin (\tau \alpha) \mathbb{D}(\hat{\mathbf{n}})+(1-\cos (\tau \alpha)) \mathbb{D}^{2}(\hat{\mathbf{n}}) . \tag{17}
\end{align*}
$$

Written like this, integration over $\tau$ is straightforward. In this case, $\mathbb{B}$ is skew-symmetric because it can be written in terms of $\mathbb{D}$, which results in $\mathrm{e}^{\tau \mathbb{B}}$ being the rotation matrix. However, other cases where $\mathbb{B}$ is not skew-symmetric can be addressed as well. For example, a force field proportional to $\mathbf{F}=(\mathbf{v} \cdot \hat{\mathbf{n}}) \hat{\mathbf{n}}-\mathbf{v}$ implies that $\mathbb{B}$ is proportional to $\mathbb{P}$, and we can use Eq. (14) to obtain a simple expression that allows us to integrate Eq. (4). Something similar can be done for other force fields, as long as $\mathbb{B}$ can be expressed as a linear combination of the operators $\mathbb{N}, \mathbb{P}, \mathbb{D}$, and $\mathbb{R}$.
In Sec. III, we will show examples where we can use these properties to find the exact solution to Eq. (4) to describe the trajectory of a particle that moves in force fields that depends on its velocity.

## III. APPLICATIONS

## A. Lorentz force

Let us consider a particle of charge $q$ that moves with a velocity $\mathbf{v}$ in the presence of an electric field $\mathbf{E}$ and a magnetic field $\mathbf{H}$, both constant in space and time. The equation of motion is governed by the Lorentz force

$$
\begin{equation*}
m \frac{\mathrm{~d} \mathbf{v}}{\mathrm{~d} t}=q(\mathbf{E}+\mathbf{v} \times \mathbf{H}) \tag{18}
\end{equation*}
$$

Defining $H \equiv\|\mathbf{H}\|, \hat{\mathbf{h}} \equiv \mathbf{H} / H$, and $\omega \equiv q H / m$, the two terms of the acceleration in Eq. (2b) in this case become $\mathbf{a}=\omega \mathbf{E} / H$ and $\mathbb{B}=\omega \mathbb{D}(\hat{\mathbf{h}})$. The trajectory of the particle is given by the solution to Eq. (4), which in this case reads

$$
\begin{align*}
\mathbf{r}(t)= & \mathbf{r}_{0}+\left(\int_{0}^{t} \mathrm{e}^{\tau \omega \mathbb{D}(\hat{\mathbf{h}})} \mathrm{d} \tau\right) \mathbf{v}_{0} \\
& +\frac{\omega}{H}\left(\int_{0}^{t} \int_{0}^{\tau} \mathrm{e}^{\xi \omega \mathbb{D}(\hat{\mathbf{h}})} \mathrm{d} \xi \mathrm{~d} \tau\right) \mathbf{E} . \tag{19}
\end{align*}
$$

If we consider an infinitesimal displacement, the previous equation reduces to

$$
\begin{align*}
d \mathbf{r}(t)= & \left(\mathbf{v}_{0}+\frac{\omega}{H} \mathbf{E} t\right) d t \\
& +\left(\mathbf{v}_{0} \times \hat{\mathbf{h}}\right) \omega t d t+(\mathbf{E} \times \hat{\mathbf{h}}) \frac{\omega^{2} t^{2}}{2 H} d t \tag{20}
\end{align*}
$$

where we have used the Taylor expansion $\mathrm{e}^{\tau \omega \mathbb{D}(\hat{\mathbf{h}})} \approx 1$ $+\tau \omega \mathbb{D}(\hat{\mathbf{h}})$ corresponding to an infinitesimal rotation about the $\hat{\mathbf{h}}$ axis (see Eq. (16)). The motion of the particle can now be understood as the sum of three contributions: a displacement of a particle moving in a constant field, $\mathbf{E}$, represented by the first term in Eq. (20); a deviation from this trajectory (second term) triggered by the magnetic field, which is
present only when the initial velocity is not null nor parallel to the magnetic field, causing the trajectory to spiral; and an accelerated motion (third term) in a direction perpendicular to both fields. Thus, the rotation operator provides an intuitive and insightful way to understand the motion of the particle.

To find the solution to Eq. (19), we have to consider that $\mathbb{D}(\hat{\mathbf{h}})$ is not invertible; therefore, the solution in Eq. (5) does not apply. However, using Eq. (16), it can be expressed as

$$
\begin{align*}
\mathbf{r}(t)= & \mathbf{r}_{0}+\left(\int_{0}^{t} \mathbb{R}(\hat{\mathbf{h}},-\omega \tau) \mathrm{d} \tau\right) \mathbf{v}_{0} \\
& +\frac{\omega}{H}\left(\int_{0}^{t} \int_{0}^{\tau} \mathbb{R}(\hat{\mathbf{h}},-\omega \xi) \mathrm{d} \xi \mathrm{~d} \tau\right) \mathbf{E} . \tag{21}
\end{align*}
$$

In this equation, we can also appreciate how the trajectory is built from successive rotations of the initial velocity vector $\mathbf{v}_{0}$ and the electric field, $\mathbf{E}$. We can solve it through direct integration after introducing Eq. (17), leading to

$$
\begin{align*}
\mathbf{r}(t)= & \mathbf{r}_{0}+\mathbf{v}_{0} t+\frac{\omega t^{2}}{2 H}(\hat{\mathbf{h}} \cdot \mathbf{E}) \hat{\mathbf{h}} \\
& +\left(\frac{1-\cos (\omega t)}{\omega}\right)\left(\left(\mathbf{v}_{0} \times \hat{\mathbf{h}}\right)-\frac{1}{H}(\mathbf{E} \times \hat{\mathbf{h}}) \times \hat{\mathbf{h}}\right) \\
& +\left(t-\frac{\sin (\omega t)}{\omega}\right)\left(\left(\mathbf{v}_{0} \times \hat{\mathbf{h}}\right) \times \hat{\mathbf{h}}+\frac{1}{H}(\mathbf{E} \times \hat{\mathbf{h}})\right) . \tag{22}
\end{align*}
$$

From this solution, it is straightforward to see that if $\mathbf{v}_{0}, \mathbf{E}$, and $\mathbf{H}$ are parallel, the particle will accelerate linearly along the $\hat{\mathbf{h}}$ axis, governed by $\mathbf{r}(t)=\mathbf{r}_{0}+\mathbf{v}_{0} t+\omega E t^{2} \hat{\mathbf{h}} / 2 H$. A helical trajectory is generated when $\mathbf{v}_{0}$ is not parallel to $\mathbf{H}$, as shown in Fig. 1, which is skewed when an electric field is present. If $\mathbf{E}$ is parallel to $\mathbf{H}$, but not to $\mathbf{v}_{0}$, the particle describes the well-known helical trajectory, given by $\mathbf{r}(t)$


Fig. 1. (Color online) A charged particle in a magnetic field $\mathbf{H}$ follows the magnetic field lines. To the left, in the absence of electric field, a particle moves in helical trajectory, following the magnetic field lines. An additional electric field $\mathbf{E}$ adds a drag to the trajectory. The color scale indicates forward evolution in time, where blue represents the initial conditions.
$=\mathbf{r}_{0}+\mathbf{v}_{0} t+\omega^{-1}(1-\cos (\omega t))\left(\mathbf{v}_{0} \times \hat{\mathbf{h}}\right)+\left(t-\omega^{-1} \sin (\omega t)\right)$ $\left(\mathbf{v}_{0} \times \hat{\mathbf{h}}\right) \times \hat{\mathbf{h}}+\left(\omega E t^{2} / 2 H\right) \hat{\mathbf{h}}$, that collapses to a circle of radius $R=v_{0} / \omega$ when $\mathbf{E}$ and $\mathbf{v}_{0}$ are perpendicular to $\mathbf{H}$. This is evident from the third term of Eq. (22), which vanishes for the latter case, implying that the motion occurs in the plane perpendicular to $\hat{\mathbf{h}}$. We can find this result in textbooks, such as Landau and Lifshitz, ${ }^{15}$ where the solution is obtained using a different approach.

## B. Coriolis force

Consider a point particle of mass $m$ that falls close to the surface of the Earth, such that the magnitude and direction of the acceleration of gravity do not change. If we neglect the air resistance and take into consideration the Coriolis force, due to the rotation of the Earth, the resulting equation of motion is

$$
\begin{equation*}
\frac{\mathrm{d} \mathbf{v}}{\mathrm{~d} t}=\mathbf{g}-2 \mathbf{\Omega} \times \mathbf{v} \tag{23}
\end{equation*}
$$

where $\boldsymbol{\Omega}$ is the constant angular velocity of the Earth and $\mathbf{g}$ is the acceleration of gravity on its surface. Examples of trajectories under this force field are shown in Fig. 2. Defining $\Omega \equiv\|\boldsymbol{\Omega}\|$ and $\hat{\boldsymbol{\omega}} \equiv \boldsymbol{\Omega} / \Omega$, we have $\mathbf{a}=\mathbf{g}$ and $\mathbb{B}=2 \Omega \mathbb{D}(\hat{\boldsymbol{\omega}})$ in Eq. (2b). The trajectory of the object can be obtained by direct analogy with the case of the particle under the Lorentz force, because the operator $\mathbb{B}$ is again proportional to $\mathbb{D}$. In this case, an infinitesimal displacement in the trajectory, given in Eq. (4), leads to
$d \mathbf{r}(t)=\left(\mathbf{v}_{0}+\mathbf{g} t\right) d t+2 \Omega\left(\mathbf{v}_{\mathbf{0}} \times \hat{\boldsymbol{\omega}}\right) t d t+(\mathbf{g} \times \hat{\boldsymbol{\omega}}) \Omega t^{2} d t$,
where we again used the Taylor expansion $\mathrm{e}^{2 \Omega \mathrm{D}(\hat{\omega})} \approx 1$ $+2 \Omega \mathbb{D}(\hat{\boldsymbol{\omega}})$, which corresponds to an infinitesimal rotation


Fig. 2. (Color online) As the Earth rotates with angular velocity $\boldsymbol{\Omega}$, an object moving to the east in the northern hemisphere is deflected to the south, due to the presence of the Coriolis force. The opposite happens in the southern hemisphere. The color scale indicates forward evolution in time, where blue represents the initial conditions. The dashed ellipses represent the origin of the trajectory. The dotted gray lines represent the projection of the trajectories on the Earth surface, while the light gray lines represent the trajectories without the Coriolis force (non-rotating Earth). A particle on the northern hemisphere, shot towards the north pole, can have a slight deflection to the west before deflecting to the east. Note: In order to allow the length of the trajectory to be non-negligible compared to the size of the Earth, these trajectories were calculated for longer paths than can be solved with high accuracy using this method, in which the direction of $\mathbf{g}$ is assumed to be constant.
about the $\hat{\omega}$ axis. As we saw in the case of a particle under the Lorentz force, we have displacement of a particle moving in a constant field, $\mathbf{g}$, a deviation in the direction perpendicular to the initial velocity, and an accelerated motion (third term) in the direction perpendicular to both $\mathbf{g}$ and $\hat{\omega}$. Replacing $\mathbb{B}=2 \Omega \mathbb{D}(\hat{\omega})$ in Eq. (4) and replacing the rotation operator from Eq. (17), we obtain

$$
\begin{align*}
\mathbf{r}(t)= & \mathbf{r}_{0}+\mathbf{v}_{\mathbf{0}} t+\frac{t^{2}}{2} \mathbf{g}+\frac{t^{2}}{2}(\mathbf{g} \times \hat{\boldsymbol{\omega}}) \times \hat{\boldsymbol{\omega}} \\
& +\left(\frac{1-\cos (2 \Omega t)}{2 \Omega}\right)\left(\left(\mathbf{v}_{0} \times \hat{\boldsymbol{\omega}}\right)-\frac{1}{2 \Omega}(\mathbf{g} \times \hat{\boldsymbol{\omega}}) \times \hat{\boldsymbol{\omega}}\right) \\
& +\left(t-\frac{\sin (2 \Omega t)}{2 \Omega}\right)\left(\left(\mathbf{v}_{0} \times \hat{\boldsymbol{\omega}}\right) \times \hat{\boldsymbol{\omega}}+\frac{1}{2 \Omega}(\mathbf{g} \times \hat{\boldsymbol{\omega}})\right) . \tag{25}
\end{align*}
$$

In textbooks, such as Mechanics by Landau and Lifshitz, only an approximation to this solution is presented, ${ }^{11}$ where terms of order $\Omega^{2}$ and higher are neglected to simplify the equations. The approximation leads to the more familiar expression for the position as a function of time

$$
\begin{equation*}
\mathbf{r}(t)=\mathbf{r}_{0}+\mathbf{v}_{0} t+\frac{1}{2} \mathbf{g} t^{2}-\frac{1}{3} \boldsymbol{\Omega} \times \mathbf{g} t^{3}-\boldsymbol{\Omega} \times \mathbf{v}_{0} t^{2} \tag{26}
\end{equation*}
$$

The general solution we present in Eq. (25) can be found in the textbook by Gantmacher, ${ }^{14}$ but the approach used to derive it is not as straightforward as ours, since it involves calculating the eigenvalues of $\mathbb{B}=2 \Omega \mathbb{D}(\hat{\boldsymbol{\omega}})$ in order to obtain an expression for $\mathrm{e}^{\tau \omega \mathbb{D}(\hat{\omega})}$ in terms of $\mathbb{D}(\hat{\boldsymbol{\omega}})$ to integrate Eq. (4).

In Fig. 3, we show how the Coriolis force changes the trajectory of a falling particle for different initial conditions at a latitude $\lambda=30^{\circ}$. Using Eq. (25), we generate a set of trajectories for angular speeds $\Omega=\Omega_{E}$ and $\Omega=2 \Omega_{E}$, where $\Omega_{E}$ is the angular speed of the Earth, by changing the initial position $\mathbf{r}_{0}$ and initial velocity $\mathbf{v}_{0}=\left(v_{x}, v_{y}, v_{z}\right)$. When $v_{z} \gg v_{x}$ and the north-south velocity $v_{y}=0$, the motion takes place primarily on the $x-z$ plane (as the normal parabolic motion does). The $y$ and $z$ components of the Coriolis force are negligible in this case. The top four panels of Fig. 3 depict the deviations from the parabolic trajectories, which are the natural paths that a projectile follows when the Coriolis force is ignored $(\Omega=0)$. When $v_{z}$ is large enough, the projectile performs a loop in the air (third panel). This can result in the particle falling in the place of departure (fourth panel), as the Earth moves towards the flying particle. In the fifth panel, a particle thrown upwards falls to the left of the departure point. The bottom three panels show how a particle in free fall is deflected to the right. To observe a few meters of deviation due to the Coriolis force, the initial height must be a few kilometers.

After observing the striking similarity between Eqs. (25) and (22), it is natural to wonder why we do not observe the same trajectories that we observed for the particle under Lorentz force. After all, Eq. (18) is the same differential equation as Eq. (23) after the assignment $q / m \rightarrow 1, \mathbf{E} \rightarrow \mathbf{g}$, $\mathbf{H} \rightarrow \mathbf{2 \Omega}$; therefore, they should lead to exactly the same solution. In particular, why do we not observe a particle subjected to the Coriolis force following a helical trajectory, but we do in the case of a charged particle in an electromagnetic field, if their trajectory is dictated by the same equation of motion? The fundamental difference relies on the fact that $\mathbf{E}$


Fig. 3. (Color online) Trajectories of a particle under the force of gravity alone (solid gray) and under gravity and deviated by the Coriolis force (colored circles). In each panel, the initial velocity is indicated by $\mathbf{v}_{0}(\mathrm{in} \mathrm{m} / \mathrm{s})$, and the initial height is either $z / z_{\max }=1$ or 0 . The same initial conditions lead to a qualitatively different trajectory (open circles) when the angular speed of the Earth, $\Omega_{E}$, is doubled. The $x$ and $y$ axes correspond to west-east and north-south directions, respectively. The $z$ axis (height) corresponds to the radial direction of the Earth.
and $\mathbf{H}$ can be chosen arbitrary and even turned off to explore different scenarios, while the gravity field and the Earth's angular velocity are much more constrained. It makes sense to study cases where $\mathbf{E}=\mathbf{0}$, but the analogous field, $\mathbf{g}$, is non-zero at every point on the surface of the Earth. Another important difference is that the direction of $\boldsymbol{\Omega}$ is fixed and the moduli of the fields $\mathbf{g}$ and $\boldsymbol{\Omega}$ do not vary.

Nevertheless, the Coriolis force produces helical trajectories, since it is a particular case of Eq. (22) with the constraints in the fields mentioned above. We can see it more clearly if we consider a particle thrown upwards at the north pole, where the gravity field and the angular speed of the Earth are parallel. In this case, $\mathbf{g} \times \hat{\boldsymbol{\omega}}=0$ and Eq. (25) becomes

$$
\begin{align*}
\mathbf{r}(t)= & \mathbf{r}_{0}+\mathbf{v}_{0} t+\frac{t^{2}}{2} \mathbf{g} \\
& +\frac{1-\cos (2 \Omega t)}{2 \Omega}\left(\mathbf{v}_{0} \times \hat{\boldsymbol{\omega}}\right) \\
& +\left(t-\frac{\sin (2 \Omega t)}{2 \Omega}\right)\left(\mathbf{v}_{0} \times \hat{\boldsymbol{\omega}}\right) \times \hat{\boldsymbol{\omega}} \tag{27}
\end{align*}
$$

This solution corresponds to a helix of radius $R=\bar{v}_{0} / 2 \Omega$, where $\bar{v}_{0}=\sqrt{v_{x}^{2}+v_{y}^{2}}$ is the modulus of the projected initial velocity on Earth's surface. This is analogous to the case of a


Fig. 4. Trajectory of a particle thrown upwards from the north pole. The initial velocity, $\mathbf{v}_{0}=(1,1,500000) \mathrm{m} / \mathrm{s}$ results in a helical trajectory of radius $R=\bar{v}_{0} / 2 \Omega$, where $\bar{v}_{0}=\sqrt{v_{x}^{2}+v_{y}^{2}}=\sqrt{2} \mathrm{~m} / \mathrm{s}$, due to the Coriolis force. The projection of the trajectory in the $\mathrm{X}-\mathrm{Y}$ plane is shown as a thick grey circle, while the trajectory that the particle would follow together with its $\mathrm{X}-\mathrm{Y}$ plane projection, are shown in solid and dashed grey lines, respectively. The maximum height reached has been normalized to one. The color scale indicates forward evolution in time, where blue represents the initial conditions.
particle in an electromagnetic field where $\mathbf{E}$ is parallel to $\mathbf{H}$. We show this trajectory in Fig. 4. It is interesting to note that a full revolution of the projected helix in the XY plane (a circular orbit) is completed only if $2 \Omega t=2 \pi$; that is, after a flight time of $T=12 \mathrm{~h}$. That is achievable only if speed in the vertical direction satisfies $v_{z}>g T / 2 \approx 211 \mathrm{~km} / \mathrm{s}$, which far exceeds the escape velocity of the Earth. Therefore, due to the slow rotation of the Earth, it is impossible to observe a particle completing a full revolution of the helical trajectory. On Jupiter, however, the Coriolis force is stronger due to the higher angular speed of the planet, which allows for the formation of the recently discovered cyclonic vortices at its poles. ${ }^{16-18}$ This is another example of how the motion of a particle can be intuitively understood as successive rotations of the initial velocity $\mathbf{v}_{0}$ and the constant field $\mathbf{a}$ that curve the trajectory.

## C. Centrifugal force

Interestingly, it is not difficult to extend the results presented here to include the centrifugal force, which appears in addition to the Coriolis force in rotating frames and depends on the instantaneous position of the particle. The magnitude of this force is of order $\Omega^{2}$, which is weaker than the Coriolis force, of order $\Omega$. Therefore, we expect that the trajectory that we derived in Sec. III B is recovered in this case when terms of order $\Omega^{2}$ are neglected.

Adding the centrifugal force to the problem from Sec. III B, the resulting equation of motion is

$$
\begin{equation*}
\frac{\mathrm{d} \mathbf{v}}{\mathrm{~d} t}=\mathbf{g}-2 \boldsymbol{\Omega} \times \mathbf{v}-\boldsymbol{\Omega} \times(\boldsymbol{\Omega} \times \mathbf{r}), \tag{28}
\end{equation*}
$$

where $\boldsymbol{\Omega}$ is the constant angular velocity of the Earth, $\mathbf{g}$ is the acceleration of gravity on its surface, and $\mathbf{r}$ is the
instantaneous position, such that $\dot{\mathbf{r}}=\mathbf{v}$. Defining $\Omega \equiv\|\Omega\|$ and $\hat{\boldsymbol{\omega}} \equiv \boldsymbol{\Omega} / \Omega$, we can write this equation as

$$
\begin{equation*}
\frac{\mathrm{d} \mathbf{v}(t)}{\mathrm{d} t}=\mathbf{a}+2 \mathbb{B} \mathbf{v}(t)+\mathbb{C} \mathbf{r}(t) \tag{29}
\end{equation*}
$$

where $\mathbf{a}=\mathbf{g}, \mathbb{B} \equiv \Omega \mathbb{D}(\hat{\boldsymbol{\omega}})$, and $\mathbb{C} \equiv \Omega^{2} \mathbb{P}(\hat{\boldsymbol{\omega}})$ are constant. Now, we introduce the change of variable

$$
\begin{equation*}
\mathbf{r}(t) \equiv \mathbb{V}(t) \mathbf{y}(t) \tag{30}
\end{equation*}
$$

where $\mathbb{V}(t)$ is defined as the operator that satisfies the differential equation $d \mathbb{V}(t) / d t=\mathbb{B}(t) \mathbb{V}(t)$ with the initial condition $\mathbb{V}(0)=\mathbb{I}$. The purpose of this transformation is to eliminate the term proportional to $\dot{\mathbf{y}}$ for equations of motion, where the operator $\mathbb{C}$ is non-zero and $\mathbb{B}(t)$ is timedependent. Thus, the velocity and acceleration satisfy the equations

$$
\begin{align*}
\dot{\mathbf{r}}(t)= & \mathbb{B}(t) \mathbb{V}(t) \mathbf{y}+\mathbb{V}(t) \dot{\mathbf{y}}(t)=\mathbf{v}(t),  \tag{31}\\
\ddot{\mathbf{r}}(t)= & \left(\dot{\mathbb{B}}(t) \mathbb{V}(t)+\mathbb{B}(t)^{2} \mathbb{V}(t)\right) \mathbf{y}(t) \\
& +2 \mathbb{B}(t) \mathbb{V}(t) \dot{\mathbf{y}}(t)+\mathbb{V}(t) \ddot{\mathbf{y}}(t) ; \tag{32}
\end{align*}
$$

therefore
$\ddot{\mathbf{y}}(t)=\mathbb{V}^{-1}(t) \mathbf{a}+\mathbb{V}^{-1}(t)\left(\mathbb{B}(t)^{2}-\dot{\mathbb{B}}(t)+\mathbb{C}\right) \mathbb{V}(t) \mathbf{y}(t)$.
In our case, $\mathbb{B}=\Omega \mathbb{D}(\hat{\boldsymbol{\omega}})$ is time independent, which means that $\mathbb{B}=0$ and $\mathbb{V}(t)=\mathrm{e}^{\mathbb{B} t}$. Then, using the property (11), Eq. (33) is reduced to

$$
\begin{equation*}
\ddot{\mathbf{y}}(t)=\mathbb{V}^{-1}(t) \mathbf{a}=\mathrm{e}^{-\Omega \mathbb{D}(\hat{\omega}) t} \mathbf{g} . \tag{34}
\end{equation*}
$$

Using the property given in Eq. (16), we obtain that $\mathbb{V}(t)$ $=\mathrm{e}^{\Omega \mathbb{D}(\hat{\omega}) t}=\mathbb{R}(\hat{\boldsymbol{\omega}},-\Omega t)$ and, therefore, $\ddot{\mathbf{y}}(t)=\mathbb{R}(\hat{\boldsymbol{\omega}}, \Omega t) \mathbf{g}$, which can be solved by direct integration using Eq. (8), leading to

$$
\begin{align*}
\mathbf{y}(t)= & \mathbf{r}_{0}+\mathbf{v}_{0} t-\Omega \mathbb{D}(\hat{\boldsymbol{\omega}}) \mathbf{r}_{0} t+\frac{t^{2}}{2} \mathbb{N}(\hat{\boldsymbol{\omega}}) \mathbf{g} \\
& -\frac{1}{\Omega^{2}} \mathbb{R}(\hat{\boldsymbol{\omega}}, \Omega t) \mathbf{g}+\frac{1}{\Omega^{2}} \mathbf{g}-\frac{t^{2}}{\Omega} \mathbb{D}(\hat{\boldsymbol{\omega}}) \mathbf{g}, \tag{35}
\end{align*}
$$

where we have used the initial conditions $\mathbf{y}(0)=\mathbb{V}^{-1}(0) \mathbf{r}(0)$ $=\mathbf{r}_{0}$ and $\dot{\mathbf{y}}(0)=\mathbf{v}_{0}-\Omega \mathbb{D}(\hat{\boldsymbol{\omega}}) \mathbf{r}_{0}$ given in Eqs. (30) and (31). Now, since $\mathbf{y}(t)=\mathbb{R}(\hat{\boldsymbol{\omega}}, \Omega t) \mathbf{r}(t)$, we can directly obtain an expression for the position, given by

$$
\begin{align*}
\mathbf{r}(t)= & \mathbf{r}_{0}+\mathbf{v}_{0} t+\frac{t^{2}}{2} \mathbf{g} \\
& -\hat{\boldsymbol{\omega}} \times\left[\left(\mathbf{r}_{0}+\mathbf{v}_{0} t+\frac{t^{2}}{2} \mathbf{g}+\frac{1}{\Omega^{2}} \mathbf{g}\right) \times \hat{\boldsymbol{\omega}}\right] \\
& +\sin (\Omega t)\left(\mathbf{r}_{0}^{*} \times \hat{\boldsymbol{\omega}}\right)+\cos (\Omega t) \hat{\boldsymbol{\omega}} \times\left(\mathbf{r}_{0}^{*} \times \hat{\boldsymbol{\omega}}\right) \\
& +\Omega t \sin (\Omega t)\left[\hat{\boldsymbol{\omega}} \times\left(\mathbf{r}_{0}^{*} \times \hat{\boldsymbol{\omega}}\right)+\frac{1}{\Omega}\left(\mathbf{v}_{0} \times \hat{\boldsymbol{\omega}}\right)\right] \\
& -\Omega t \cos (\Omega t)\left[\left(\mathbf{r}_{0}^{*} \times \hat{\boldsymbol{\omega}}\right)-\frac{1}{\Omega} \hat{\boldsymbol{\omega}} \times\left(\mathbf{v}_{0} \times \hat{\boldsymbol{\omega}}\right)\right], \tag{36}
\end{align*}
$$

where $\mathbf{r}_{0}^{*} \equiv \mathbf{r}_{0}+\left(1 / \Omega^{2}\right) \mathbf{g}$. It is not difficult to verify that Eq. (26) is recovered when terms of order $\Omega^{2}$ are neglected.

Interestingly, it is, in principle, possible to solve this problem when there is a variable angular velocity as well, that is $\boldsymbol{\Omega}=\omega(t) \hat{\boldsymbol{\omega}}$. If the explicit form of the speed $\omega(t)$ is simple, the integrals can be evaluated and an analytical solution can be found. ${ }^{3}$

## D. Lift and friction force

Let us consider the motion of a free-falling sheet of paper, where the air resistance exerts a drag and lift forces such that the resulting force acts perpendicular to the velocity. ${ }^{19,20}$ In addition, let us consider that the magnitude of such a force is proportional to the velocity and a more general case, where the direction of the total force is not necessarily perpendicular to the velocity, can be considered. The equation of motion that models such situation is

$$
\begin{equation*}
m \frac{\mathrm{~d} \mathbf{v}}{\mathrm{~d} t}(t)=m \mathbf{a}-k \mathbb{R}(\hat{\mathbf{n}}, \varphi) \mathbf{v}(t) \tag{37}
\end{equation*}
$$

where $\mathbb{R}(\hat{\mathbf{n}}, \varphi)$ is the rotation operator defined in Eq. (6) and $k$ is a constant representing the viscous friction coefficient. When the lift force is substantially greater than the drag force, the force is approximately perpendicular to the velocity. Comparison with Eq. (2b) yields $\mathbb{B}=-\kappa \mathbb{R}(\hat{\mathbf{n}}, \varphi)$, where $\kappa \equiv k / m$. From Eq. (4), the trajectory of the particle can be found by solving

$$
\begin{align*}
\mathbf{r}(t)= & \mathbf{r}_{0}+\left(\int_{0}^{t} \mathrm{e}^{-\kappa \tau \mathbb{R}(\hat{\mathbf{n}}, \varphi)} \mathrm{d} \tau\right) \mathbf{v}_{0} \\
& +\left(\int_{0}^{t} \int_{0}^{\tau} \mathrm{e}^{-\kappa \xi \mathbb{R}(\hat{\mathbf{n}}, \varphi)} \mathrm{d} \xi \mathrm{~d} \tau\right) \mathbf{a} . \tag{38}
\end{align*}
$$

In this case, the inverse of $\mathbb{B}$ does exist and the solution is provided in Eq. (5)

$$
\begin{align*}
\mathbf{r}(t)= & \mathbf{r}_{0}+t \kappa^{-1} \mathbb{R}(\hat{\mathbf{n}},-\varphi) \mathbf{a}+\kappa^{-1} \mathbb{R}(\hat{\mathbf{n}},-\varphi) \\
& \times\left(\mathbb{I}-\mathrm{e}^{-t \kappa \mathbb{R}(\hat{\mathbf{n}}, \varphi)}\right)\left(\mathbf{v}_{0}-\kappa^{-1} \mathbb{R}(\hat{\mathbf{n}},-\varphi) \mathbf{a}\right) . \tag{39}
\end{align*}
$$

Using the properties of the matrices shown in Sec. II B, it is possible to rewrite the term $\mathrm{e}^{-t \kappa \mathbb{R}(\hat{\mathbf{n}}, \varphi)}$ (see Appendix B) to attain a more manageable expression, such that the equation above can be written as

$$
\begin{align*}
\mathbf{r}(t)= & \mathbf{r}_{0}+\frac{t}{\kappa} \mathbb{R}(\hat{\mathbf{n}},-\varphi) \mathbf{a}+\frac{\mathbb{R}(\hat{\mathbf{n}},-\varphi)}{\kappa^{2}}\left(\kappa \mathbf{v}_{0}-\mathbb{R}(\hat{\mathbf{n}},-\varphi) \mathbf{a}\right) \\
& -\frac{\mathrm{e}^{-\kappa t \cos \varphi}}{\kappa^{2}} \mathbb{R}(\hat{\mathbf{n}},-\varphi-\kappa t \sin \varphi)\left(\kappa \mathbf{v}_{0}-\mathbb{R}(\hat{\mathbf{n}},-\varphi) \mathbf{a}\right) \\
& -\frac{\left(\mathrm{e}^{-\kappa t}-\mathrm{e}^{-\kappa t \cos \varphi}\right)}{\kappa^{2}} \mathbb{N}(\hat{\mathbf{n}})\left(\kappa \mathbf{v}_{0}-\mathbf{a}\right) \tag{40}
\end{align*}
$$

To gain a deeper physical insight of this solution, we can consider a differential displacement in Eq. (38), using the approximation $\mathrm{e}^{-\kappa \tau \mathbb{R}(\hat{\boldsymbol{n}}, \varphi)} \approx 1-\kappa \tau \mathbb{R}(\hat{\boldsymbol{n}}, \varphi)$, which leads to

$$
\begin{align*}
d \mathbf{r}(t)= & \left(\mathbf{v}_{0}+\mathbf{a} t\right) d t \\
& -\left(\kappa \mathbb{R}(\hat{\boldsymbol{n}}, \varphi) \mathbf{v}_{0}\right) t d t-(\kappa \mathbb{R}(\hat{\boldsymbol{n}}, \varphi) \mathbf{a}) \frac{t^{2}}{2} d t . \tag{41}
\end{align*}
$$

As we observed in the previous cases of the Coriolis and Lorentz force, we again have a contribution from acceleration in a constant field in the first term. However, the other contributions are not perpendicular to the initial velocity, $\mathbf{v}_{0}$,


Fig. 5. Trajectories of a particle falling under the force of gravity $(\mathbf{a}=-g \hat{\mathbf{y}})$ and drag and lift forces, generated in Eq. (40). In each panel, we show a different force, which is obtained by rotating the instantaneous velocity of the particle an angle $\varphi$ about the $\hat{\mathbf{z}}$ axis. Different line styles represent different friction coefficients $\kappa=k / m$.
and to the constant field $\mathbf{a}$, as in the previous cases. Instead, the operator $\mathbb{R}(\hat{\mathbf{n}}, \varphi)$ causes an infinitesimal displacement in the direction of the rotated fields with angle $\varphi$.

Note that when $\varphi=0$ in Eq. (40), $\mathbb{R}(\hat{\mathbf{n}},-\varphi)=\mathbb{I}$ and we recover the usual result of a lift force acting parallel to the velocity, namely,

$$
\begin{equation*}
\mathbf{r}(t)=\mathbf{r}_{0}+\frac{t}{\kappa} \mathbf{a}+\frac{1-\mathrm{e}^{-\kappa t}}{\kappa^{2}}\left(\kappa \mathbf{v}_{0}-\mathbf{a}\right), \tag{42}
\end{equation*}
$$

which represents, for example, the trajectory of a particle falling in a viscous fluid if $m \mathbf{a}$ is the difference between the weight and buoyancy, or the projectile motion of a point mass with air resistance.

For $\varphi= \pm \pi / 2$ and $\mathbf{a}=\mathbf{g}$, Eq. (37) describes the motion of a sheet of paper in free fall, where the air resistance acts perpendicular to the velocity of the object. To compare with the result in Ref. 19, we will consider $\mathbf{r}_{0}=\mathbf{0}=\mathbf{v}_{0}$ and that the motion takes place on the $X-Y$ plane, where the force of gravity is excerted along the $Y$ axis. The direction of the air resistance is given by a rotation of $\varphi= \pm \pi / 2$ about the $\hat{\mathbf{n}}=\hat{\mathbf{z}}$ axis and depends on the direction of the displacement (left or right). The different sections of the motion, after replacing these values in Eq. (40), are described by

$$
\begin{align*}
\mathbf{r}(t)= & \frac{t}{\kappa} \mathbb{R}\left(\hat{\mathbf{z}}, \mp \frac{\pi}{2}\right) \mathbf{g} \\
& -\frac{1}{\kappa^{2}} \mathbb{R}(\hat{\mathbf{z}}, \mp \pi) \mathbf{g}+\frac{1}{\kappa^{2}} \mathbb{R}(\hat{\mathbf{z}}, \mp(\pi+\kappa t)) \mathbf{g} \tag{43}
\end{align*}
$$

where we have used $\mathbb{N}(\hat{\mathbf{z}}) \mathbf{v}_{0}=\mathbb{N}(\hat{\mathbf{z}}) \mathbf{g}=\mathbf{0}$. Replacing the acceleration of gravity $\mathbf{a}=-g \hat{\mathbf{y}}$, we obtain

$$
\begin{equation*}
\mathbf{r}(t)= \pm \frac{g}{\kappa^{2}}(\sin (\kappa t)-\kappa t) \hat{\mathbf{x}}+\frac{g}{\kappa^{2}}(\cos (\kappa t)-1) \hat{\mathbf{y}} \tag{44}
\end{equation*}
$$

This model, of course, does not incorporate all of the complexity of the real case. ${ }^{20,21}$ However, more realistic solutions are obtained by considering an angle $\varphi> \pm \pi / 2$.

In Fig. 5, we show different trajectories of a particle experiencing a lift force $-k \mathbb{R}(\hat{\mathbf{z}}, \varphi) \mathbf{v}(t)$ and its own weight. When $\varphi=\pi / 4$, the particle is deflected to the left, which corresponds to the direction of the force. For $\varphi=\pi / 2$, the force acts perpendicular to the velocity, which deflects the particle upwards periodically, reaching its initial height every period. This corresponds to the simplified model of a falling sheet of paper. A lift force that forms an angle higher than $90^{\circ}$ with the velocity (two panels at the bottom) generates loops in the air that become wider as the angle increases.

## IV. CONCLUSIONS

Our proposed method provides exact solutions to the equations of motion of a particle moving in three dimensions, subjected to forces that depend linearly on its instantaneous velocity and position, and it represents a viable alternative to other solution techniques such as matrix diagonalization. We exploit properties of the rotation matrix and its generators. The formal solution to Eq. (2) can be easily extended to the case in which $\mathbb{B}$ depends on time, as long as $\mathbb{B}=\mathbb{B}(t)$ at a given instant of time commutes with the value $\mathbb{B}$ at any other instant of time. If it does not commute, it is necessary to appeal to the Magnus expansion. ${ }^{22}$ Other dynamical systems, where the force field can be expressed as a linear combination of the operators $\mathbb{N}, \mathbb{P}, \mathbb{D}$, and $\mathbb{R}$, can also be addressed within this approach.

We presented an explicit solution to a number of seemingly unrelated problems: a particle under the influence of Lorentz force, the Coriolis and centrifugal forces, and a generalized lift force. These applications can be easily extended within this approach to other interesting cases such as a particle in a timedependent magnetic field, the classical Hall effect, a relativistic particle in an electric field, harmonic oscillators with dissipative terms, normal modes, and many others.

Although not all problems that can be written as Eq. (2) may be solved within our approach, and diagonalization may be unavoidable in some cases, it is certainly useful for a large number of them. The presented approach can also be used to find an exact solution to a series of problems that have been addressed from different approaches that can be modeled by a differential equation, where an operator applied to a vector is equal to its derivative. ${ }^{2-10}$ Revisiting these problems can provide an insightful view into what Feynman referred to as the underlying unity of nature.

## ACKNOWLEDGMENTS

G.G. thanks partial support from Fondecyt 1171127. D.L. acknowledges partial financial support from FONDECYT 1180905.

## APPENDIX A: RODRIGUES' ROTATION FORMULA AND ROTATION MATRICES

## 1. Rotation of a vector

Let $\varphi \in[0,2 \pi)$ and $\hat{n} \in \mathbb{R}^{3}$ a unitary vector $(\|\hat{n}\|=1)$. Let $\mathbb{R}(\hat{n}, \varphi)$ be the rotation transformation that, as Fig. 6 shows, takes the vector $\vec{x}$ to $\vec{x}^{\prime}$ through


Fig. 6. Rotation in $\varphi$ radians about the axis $\hat{n}$.

$$
\begin{equation*}
\vec{x}^{\prime}=\mathbb{R}(\hat{n}, \varphi) \vec{x} \tag{A1}
\end{equation*}
$$

The vector $\vec{x}$ can be separated into longitudinal and transverse components with respect to the rotation axis $\hat{n}$, as shown in Fig. 7

$$
\begin{equation*}
\vec{x}=\vec{x}_{L}+\vec{x}_{T}, \tag{A2}
\end{equation*}
$$

where

$$
\begin{equation*}
\vec{x}_{L} \equiv(\vec{x} \cdot \hat{n}) \hat{n} \tag{A3}
\end{equation*}
$$

and

$$
\begin{align*}
\vec{x}_{T} & \equiv \vec{x}-\vec{x}_{L} \\
& =(\hat{n} \cdot \hat{n}) \vec{x}-(\hat{n} \cdot \vec{x}) \hat{n} \\
& =\hat{n} \times(\vec{x} \times \hat{n}) . \tag{A4}
\end{align*}
$$

Then,

$$
\begin{equation*}
\vec{x}=(\vec{x} \cdot \hat{n}) \hat{n}+\hat{n} \times(\vec{x} \times \hat{n}) . \tag{A5}
\end{equation*}
$$



Fig. 7. Longitudinal and transverse components of $\vec{x}$.

From Fig. 7, we can conclude that

$$
\begin{equation*}
\vec{x}_{L}^{\prime}=\vec{x}_{L} \tag{A6}
\end{equation*}
$$

and

$$
\begin{equation*}
\vec{x}_{T}^{\prime}=\vec{x}_{T} \cos (\varphi)+(\hat{n} \times \vec{x}) \sin (\varphi) . \tag{A7}
\end{equation*}
$$

Then, we can infer that

$$
\begin{align*}
\vec{x}^{\prime}= & \mathbb{R}(\hat{n}, \varphi) \vec{x} \\
= & (\hat{n} \cdot \vec{x}) \hat{n}+\hat{n} \times(\vec{x} \times \hat{n}) \cos (\varphi) \\
& +(\hat{n} \times \vec{x}) \sin (\varphi) . \tag{A8}
\end{align*}
$$

## 2. Rotation matrices

We can define the transformations $\mathbb{N}, \mathbb{P}$, and $\mathbb{D}$ as

$$
\begin{align*}
& \mathbb{N}(\hat{n}) \vec{x}=(\hat{n} \cdot \vec{x}) \hat{n},  \tag{A9a}\\
& \mathbb{P}(\hat{n}) \vec{x}=\hat{n} \times(\vec{x} \times \hat{n}),  \tag{A9b}\\
& \mathbb{D}(\hat{n}) \vec{x}=\vec{x} \times \hat{n} . \tag{A9c}
\end{align*}
$$

From Eq. (A8), it is clear that

$$
\begin{equation*}
\mathbb{R}(\hat{n}, \varphi)=\mathbb{N}(\hat{n})+\cos (\varphi) \mathbb{P}(\hat{n})-\sin (\varphi) \mathbb{D}(\hat{n}) \tag{A10}
\end{equation*}
$$

We notice that if $\hat{n}=\left(n_{1}, n_{2}, n_{3}\right)$, then the corresponding transformation matrices are

$$
\begin{array}{ll}
{[\mathbb{N}(\hat{n})]_{i j}=n_{i} n_{j}} & (\text { projector } \|), \\
{[\mathbb{P}(\hat{n})]_{i j}=\delta_{i j}-n_{i} n_{j}} & (\text { projector } \perp), \\
{[\mathbb{D}(\hat{n})]_{i j}=\epsilon_{i j k} n_{k}} & (\text { dual of } \hat{n}) \tag{A13}
\end{array}
$$

In explicit matrix form,

$$
\begin{align*}
& \mathbb{N}(\hat{n})=\left[\begin{array}{ccc}
n_{1}^{2} & n_{1} n_{2} & n_{1} n_{3} \\
n_{1} n_{2} & n_{2}^{2} & n_{2} n_{3} \\
n_{1} n_{3} & n_{2} n_{3} & n_{3}^{2}
\end{array}\right],\left[\begin{array}{ccc}
1-n_{1}^{2} & -n_{1} n_{2} & -n_{1} n_{3} \\
-n_{1} n_{2} & 1-n_{2}^{2} & -n_{2} n_{3} \\
-n_{1} n_{3} & -n_{2} n_{3} & 1-n_{3}^{2}
\end{array}\right],  \tag{A14a}\\
& \mathbb{P}(\hat{n})=\left[\begin{array}{ccc}
0 & n_{3} & -n_{2} \\
-n_{3} & 0 & n_{1} \\
n_{2} & -n_{1} & 0
\end{array}\right] . \tag{A14b}
\end{align*}
$$

It is easy to verify some properties of these matrices, such as

$$
\begin{equation*}
\mathbb{I}=\mathbb{P}+\mathbb{N} \tag{A15}
\end{equation*}
$$

and that their squares satisfy

$$
\begin{align*}
& \mathbb{N}^{2}=\mathbb{N}  \tag{A16a}\\
& \mathbb{P}^{2}=\mathbb{P} \tag{A16b}
\end{align*}
$$

$$
\begin{align*}
& \mathbb{D}^{2}=-\mathbb{P}  \tag{A16c}\\
& \mathbb{D}^{2}=-\mathbb{P}  \tag{A16d}\\
& \mathbb{P}+\mathbb{N}=\mathbb{I} \tag{A16e}
\end{align*}
$$

In addition, they all commute with each other and are non-invertible. Using the previous properties, we can verify that

$$
\begin{equation*}
\mathbb{R}(\hat{n}, \varphi)=\mathbb{I}-\sin (\varphi) \mathbb{D}(\hat{n})+(1-\cos (\varphi)) \mathbb{D}^{2}(\hat{n}) . \tag{A17}
\end{equation*}
$$

Using the Maclaurin series of the exponential, it is easy to demonstrate that

$$
\begin{equation*}
\mathbb{R}(\hat{n}, \varphi)=\mathrm{e}^{-\varphi \mathbb{D}(\hat{n})} . \tag{A18}
\end{equation*}
$$

This last result shows that $\mathbb{D}$ is the generator of $\mathbb{R}$.
The properties listed above can be used to demonstrate that these operators are also related by the following identities:

$$
\begin{align*}
& \mathrm{e}^{\alpha \mathbb{N}(\hat{\mathbf{n}})}=\mathbb{P}(\hat{\mathbf{n}})+\mathrm{e}^{\alpha} \mathbb{N}(\hat{\mathbf{n}}),  \tag{A19a}\\
& \mathrm{e}^{\beta \mathbb{P}(\hat{\mathbf{n}})}=\mathbb{N}(\hat{\mathbf{n}})+\mathrm{e}^{\beta} \mathbb{P}(\hat{\mathbf{n}}),  \tag{A19b}\\
& \mathrm{e}^{-\varphi \mathbb{D}(\hat{\mathbf{n}})}=\mathbb{N}(\hat{\mathbf{n}})+\cos \varphi \mathbb{P}(\hat{\mathbf{n}})-\sin \varphi \mathbb{D}(\hat{\mathbf{n}}),  \tag{A19c}\\
& \mathrm{e}^{\alpha \mathbb{R}(\hat{\mathbf{n}}, \varphi)}= \\
& =\mathrm{e}^{\alpha \mathbb{N}(\hat{\mathbf{n}})} \mathrm{e}^{\alpha \cos \varphi \mathbb{P}(\hat{\mathbf{n}})} \mathrm{e}^{-\alpha \sin \varphi \mathbb{D}(\hat{\mathbf{n}})} \\
& = \\
& =\mathrm{e}^{\alpha \mathbb{N}(\hat{\mathbf{n}})} \mathrm{e}^{\alpha \cos \varphi \mathbb{P}(\hat{\mathbf{n}}) \mathbb{R}(\hat{\mathbf{n}}, \alpha \sin \varphi),}  \tag{A19d}\\
& = \\
& =\mathrm{e}^{\alpha} \mathbb{N}(\hat{\mathbf{n}})+\mathrm{e}^{\alpha \cos \varphi} \cos (\alpha \sin \varphi) \mathbb{P}(\hat{\mathbf{n}}) \\
& \quad-\mathrm{e}^{\alpha \cos \varphi} \sin (\alpha \sin \varphi) \mathbb{D}(\hat{\mathbf{n}}), \\
& \mathrm{e}^{\alpha \mathbb{N}(\hat{\mathbf{n}})} \mathrm{e}^{\beta \mathbb{P}(\hat{\mathbf{n}})}=\mathrm{e}^{\alpha \mathbb{N}(\hat{\mathbf{n}})+\mathrm{e}^{\beta} \mathbb{P}(\hat{\mathbf{n}}) .}
\end{align*}
$$

(A19e)

## APPENDIX B: THE OPERATOR $\mathrm{e}^{-t \kappa} \mathbb{R}(\hat{\mathbf{n}}, \varphi)$

$\underset{-t \kappa \mathbb{R}(\mathbf{n}, \varphi)}{\text { Using }}$ as. (15) and (16), we can rewrite the operator $\mathrm{e}^{-t \kappa \mathbb{R}(\hat{\mathbf{n}}, \varphi)}$ as

$$
\begin{align*}
\mathrm{e}^{-t \kappa \mathbb{R}(\hat{\mathbf{n}}, \varphi)} & =\mathrm{e}^{-t \kappa \mathbb{I}} \mathrm{e}^{t \kappa \sin \varphi \mathbb{D}(\hat{\mathbf{n}})} \mathrm{e}^{-t \kappa(1-\cos \varphi) \mathbb{D}^{2}(\hat{\mathbf{n}})} \\
& =\mathrm{e}^{-t \kappa} \mathbb{R}(\hat{\mathbf{n}},-t \kappa \sin \varphi) \mathrm{e}^{-t \kappa(1-\cos \varphi) \mathbb{D}^{2}(\hat{\mathbf{n}})} \tag{B1}
\end{align*}
$$

Since $\mathrm{e}^{c \mathbb{D}^{2}(\hat{\mathbf{n}})}=\mathrm{e}^{-c \mathbb{P}(\hat{\mathbf{n}})}=\mathbb{N}(\hat{\mathbf{n}})+\mathrm{e}^{-c} \mathbb{P}(\hat{\mathbf{n}})$, we have

$$
\begin{equation*}
\mathrm{e}^{-t \kappa \mathbb{R}(\hat{\mathbf{n}}, \varphi)}=\mathrm{e}^{-t \kappa} \mathbb{R}(\hat{\mathbf{n}},-t \kappa \sin \varphi)\left[\mathbb{N}(\hat{\mathbf{n}})+\mathrm{e}^{-c} \mathbb{P}(\hat{\mathbf{n}})\right], \tag{B2}
\end{equation*}
$$

where $c=-t \kappa(1-\cos \varphi)$. Substituting the equation above in Eq. (39), we obtain

$$
\begin{align*}
\mathbf{r}(t)= & \mathbf{r}_{0}+t \kappa^{-1} \mathbb{R}(\hat{\mathbf{n}},-\varphi) \mathbf{a}+\kappa^{-1}[\mathbb{R}(\hat{\mathbf{n}},-\varphi) \\
& -\mathrm{e}^{-t \kappa} \mathbb{R}(\hat{\mathbf{n}},-\varphi-t \kappa \sin \varphi)(\mathbb{N}(\hat{\mathbf{n}}) \\
& \left.\left.+\mathrm{e}^{t \kappa(1-\cos \varphi)} \mathbb{P}(\hat{\mathbf{n}})\right)\right]\left(\mathbf{v}_{0}-\kappa^{-1} \mathbb{R}(\hat{\mathbf{n}},-\varphi) \mathbf{a}\right) . \tag{30}
\end{align*}
$$

Using the properties in Eqs. (9)-(15), we can redistribute the terms in Eq. (B3), which leads to

$$
\begin{align*}
\mathbf{r}(t)= & \mathbf{r}_{0}+\frac{t}{\kappa} \mathbb{R}(\hat{\mathbf{n}},-\varphi) \mathbf{a}+\frac{\mathbb{R}(\hat{\mathbf{n}},-\varphi)}{\kappa^{2}}\left(\kappa \mathbf{v}_{0}-\mathbb{R}(\hat{\mathbf{n}},-\varphi) \mathbf{a}\right) \\
& -\frac{\mathrm{e}^{-\kappa t \cos \varphi}}{\kappa^{2}} \mathbb{R}(\hat{\mathbf{n}},-\varphi-\kappa t \sin \varphi)\left(\kappa \mathbf{v}_{0}-\mathbb{R}(\hat{\mathbf{n}},-\varphi) \mathbf{a}\right) \\
& -\frac{\left(\mathrm{e}^{-\kappa t}-\mathrm{e}^{-\kappa t \cos \varphi}\right)}{\kappa^{2}} \mathbb{N}(\hat{\mathbf{n}})\left(\kappa \mathbf{v}_{0}-\mathbf{a}\right), \tag{B4}
\end{align*}
$$

which is the result shown in Eq. (40).
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