

# Thermal convection in a rotating binary viscoelastic liquid mixture

D. Laroze<sup>1</sup>, J. Martínez-Mardones<sup>1</sup>, and J. Bragard<sup>2,a</sup>

<sup>1</sup> Instituto de Física, Pontificia Universidad Católica de Valparaíso, Casilla 4059, Valparaíso, Chile

<sup>2</sup> Departamento de Física y Matemática Aplicada, Universidad de Navarra, 31080 Pamplona, España

**Abstract.** In this work we report theoretical and numerical results on convection in a viscoelastic binary mixture under rotation. In particular, we focus in the Maxwellian case of viscoelastic fluid. We obtain explicit expressions for the convective thresholds in terms of the mixture parameters of the system in the case of idealized boundary conditions. We also calculate numerically the convective thresholds for the case of realistic rigid–rigid boundary conditions.

## 1 Introduction

A particularly interesting technique in molecular biology is the replication of DNA in a chain reaction [1]. Usually, DNA fragments mixed with thermostable polymerase enzyme are heated until the temperature required to unbound the double helix is reached (above 80 °C). Then the mixture is annealed, and DNA fragments are doubled by the polymerase enzyme. After several thermal cycles have been completed a high concentration of DNA is obtained. This is known as polymerase chain reaction (PCR), a standard technique usually found in biochemistry laboratories [1]. Thermal cyclers in the standard PCR heat and cool not only the whole reaction volume, but also the surrounding vessel and its fitting. This implies a large thermal mass involved which delays heating and cooling. Recently Krishna et al. [2] proposed to replace conventional thermocyclers by Rayleigh–Bénard convection cells. The same year, Braun and Libchaber [3] found an efficient trapping mechanism of DNA in solution through the interplay of convection and thermophoresis. In a subsequent article, Braun et al. [4] showed that PCR may be driven by the temperature difference of a laminar convection cell. This convective PCR is several times faster than the conventional one since only the temperature of the liquid is varied and the vessel is kept at a constant temperature. To improve further this new technique based on thermal convection, it is desirable to explore in detail the different regimes of convection in DNA suspensions.

Convection in aqueous suspensions of DNA is not a trivial subject. First of all, these suspensions are binary and the Soret effect, that couples thermal and concentrational gradients, is important (see Cross and Hohenberg [5]). Binary fluids subjected to a vertical temperature gradient experience both thermal and solutal stratification. Due to the fact that the fluid density depends on solute concentration, it leads to a competition between thermal and compositional gradients. This competition between heat diffusion and solute diffusion may lead to oscillations in the layer. The oscillatory convective instability in binary fluid mixtures is adequately understood in quantitative detail, both experimentally and theoretically [5,6]. A second important effect that has to be considered while working with DNA suspensions is that under the action of shear or of externally-applied tension, the DNA coils can become extended. Consequently viscoelastic properties of DNA must be taken into account in realistic models of DNA

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<sup>a</sup> e-mail: jbragard@fisica.unav.es

convection. Chu and collaborators [9–12] have performed a series of experiments in order to determine the behavior of single DNA molecules under shear stresses. The longest time scale for the relaxation of a DNA molecule was found to be as large as few seconds, comparable with the thermal diffusion time scale  $\tau = d^2/\kappa$  which is order of 1 s. This demonstrates that viscoelastic properties are relevant in dealing with convection in DNA suspensions. Kolodner [7] reported observations of oscillatory convection in DNA suspensions. The DNA concentrations that were used in his experiments ranged from dilute to semi-dilute. The same author convincingly proved that viscoelasticity is the cause of the oscillations obtained in his experiments.

Theoretical convective thresholds for binary mixtures [5,6], viscoelastic fluids [13], and binary viscoelastic fluids [14] can be found in the literature. It has been demonstrated that instability may be stationary and oscillatory, depending on the binary and viscoelastic properties of the fluid. Rotation of the entire fluid vessel is another important effect that can modify the convective regime. It is known from Chandrasekhar [15] that in general rotation has a stabilizing effect on convection. Rotating convection in Newtonian fluid [16], binary mixture [17–19], and viscoelastic fluids [20] have been thoroughly studied.

The purpose of the present paper is to analyze the influence of rotation in convective thresholds in binary viscoelastic mixtures and, in particular, in aqueous DNA suspensions. To this aim a binary mixture of Oldroyd viscoelastic fluid is heated from below and rotated around the vertical axis. The description of the system involves many parameters whose values have not yet been determined accurately. Therefore, we are left with some freedom in fixing the parameter values. In order to be as exhaustive as possible, we will analyze the linear regime for two different limiting cases of boundary condition i.e. the free–free and the rigid–rigid boundary conditions. In the first case (free–free), one can explicitly calculate the threshold for convection in function of the parameters of the fluid mixture. In addition, we have further checked that we retrieved some old results for simplified situations obtained perviously by other authors. In the case of realistic boundary conditions (rigid–rigid), an analytical calculation is not tractable and we numerically solve the system using a spectral method in order to determine the eigenfunctions and eigenvalues (the convective thresholds). The paper is organized as follows. In section 2, the basic hydrodynamic equations for binary viscoelastic convection are presented. In section 3, the linear stability analysis of the conduction state is performed, and the conditions for the onset of convection are discussed. Finally, conclusions are drawn in section 4.

## 2 Balance and constitutive equations

We consider a layer of incompressible binary viscoelastic fluid, of thickness  $d$  and very large horizontal extension, in a gravitational field and submitted to a vertical temperature gradient. It is assumed that the layer is rotating uniformly about the vertical axis with an angular velocity  $\varpi$ . In the Boussinesq approximation, the balance equations can be written [16]:

$$\nabla \cdot \mathbf{v} = 0 \quad (1)$$

$$\rho_0(\partial_t + \mathbf{v} \cdot \nabla)\mathbf{v} = -\nabla p + \nabla \cdot \boldsymbol{\tau} + \rho \mathbf{g} + 2\rho_0\varpi \times \mathbf{v} + \frac{1}{2}\rho_0\nabla(|\varpi \times \mathbf{r}|^2) \quad (2)$$

$$(\partial_t + \mathbf{v} \cdot \nabla)T = \kappa\nabla^2 T \quad (3)$$

$$(\partial_t + \mathbf{v} \cdot \nabla)N = D(\nabla^2 N + (k_T/T_0)\nabla^2 T), \quad (4)$$

where  $\mathbf{v} = (u, v, w)^T$  is the velocity field,  $\rho$  the mixture density,  $p$  the pressure,  $\boldsymbol{\tau}$  the extra stress tensor,  $T$  the temperature,  $N$  the polymer concentration,  $\mathbf{r}$  the position vector,  $\mathbf{g}$  the gravity acceleration,  $\kappa$  the thermal diffusivity,  $D$  the solute diffusivity, and  $k_T$  the Soret coefficient.  $T_0$  and  $\rho_0$  denote reference values of temperature and density, respectively. For the liquid mixture, the following state equation is used:  $\rho = \rho_0[1 - \alpha(T - T_0) + \beta(N - N_0)]$ , where  $\alpha$  and  $\beta$  are thermal and mass expansion coefficients, respectively and  $N_0$  is a reference value for polymeric concentration.

In addition to the above governing equations, a constitutive equation relating the extra stress tensor  $\boldsymbol{\tau}$  and the shear rate has to be added in order to be able to solve the system of

equations (1)–(4). For a simple (Newtonian) fluid, this constitutive equation reduces to  $\tau = 2\nu\mathbf{D}$ , where  $\nu$  is the kinematic viscosity and  $\mathbf{D}$  is the symmetric part of velocity field gradient which is a tensor of rank two [21]. For complex fluids, the constitutive relation between stress and shear is by far more involved. There exist many models for characterizing these fluids, one class of such fluids is described by the Oldroyd model [8]:  $(1 + \lambda_1 D_t)\tau = 2\nu(1 + \lambda_2 D_t)\mathbf{D}$ , in which  $\nu$  is the static viscosity,  $\lambda_1$  is the relaxation time and  $\lambda_2$  is the retardation time, parameters that characterize the viscoelastic properties. The symbol  $D_t$  denotes an invariant (frame-indifferent) time derivative, defined as  $D_t\tau = \partial_t\tau + (\mathbf{v}\nabla\cdot)\tau + \tau\cdot\mathbf{W} - \mathbf{W}\cdot\tau + a(\tau\cdot\mathbf{D} + \mathbf{D}\cdot\tau)$ , where  $\mathbf{W}$  is the skew-symmetric part of the velocity field gradient, and  $a$  is a phenomenological parameter in the range  $[-1, 1]$ . For  $a = -1$ , one gets the lower convected Jeffrey's model (Oldroyd B),  $a = 0$  leads to the so-called corotational Jeffrey's model, and  $a = 1$  corresponds to the upper convected Jeffrey's model (Oldroyd A). Let us fix the  $z$ -axis such that  $\mathbf{g} = -g\hat{\mathbf{z}}$  and the liquid layer is comprised between the two horizontal planes at  $z = 0$  and  $z = d$ . The boundary conditions for the temperature are  $T(z = 0) = T_0 + \Delta T$ ,  $T(z = d) = T_0$  (boundaries are thermal conductors) and for the polymeric concentration, one imposes that  $\partial_z[N + (k_T/T_0)T] = 0$  at both boundaries which corresponds to impermeable frontiers. For the vertical velocity, we first consider the unrealistic conditions:  $w = \partial_z^2 w = 0$  (free-free) and subsequently, the second case:  $w = \partial_z w = 0$  (rigid-rigid) which is realistic but less tractable analytically. The reference motionless solution (it is assumed that the viscoelastic liquid has relaxed enough time, typically one second is enough for dilute polymeric suspensions [8]) from which we will study the stability is easily obtained, one gets:

$$\mathbf{v}_c = \mathbf{0} \quad (5)$$

$$\tau_c = \mathbf{0} \quad (6)$$

$$T_c(z) = T_0 + \Delta T(1 - z/d) \quad (7)$$

$$N_c(z) = N_0 + \Delta N(1 - z/d) \quad (8)$$

$$\nabla(p + |\varpi \times \mathbf{r}|^2/2) = -g\rho_0(1 - \alpha\Delta T + \beta\Delta N)(1 - z/d), \quad (9)$$

where  $\Delta N = -(k_T/T_0)\Delta T$ . Before performing the linear stability analysis of the conducting state defined by equations (5)–(9), it is useful to rescale the variables by  $d$  (length),  $d^2/\kappa$  (time),  $\kappa/d$  (velocity),  $\nu\kappa/d^2$  (stress tensor),  $\Delta T$  (temperature),  $\Delta N$  (concentration). In the dimensionless equations,  $\theta$  and  $c$  denote the temperature and concentration, respectively. It is useful for simplifying the calculations to define a new variable  $\eta = c - \theta$ . The dimensionless equations for the perturbations around the conducting state read as:

$$\nabla \cdot \mathbf{v} = 0 \quad (10)$$

$$\mathbf{P}^{-1}(\partial_t + \mathbf{v} \cdot \nabla)\mathbf{v} = -\nabla p + \nabla \cdot \tau + [(1 + \psi)\theta + \psi\eta]\hat{\mathbf{z}} + \sqrt{T_a}(\mathbf{v} \times \hat{\mathbf{z}}) \quad (11)$$

$$(\partial_t + \mathbf{v} \cdot \nabla)\theta = \text{Ra} w + \nabla^2 \theta \quad (12)$$

$$(\partial_t + \mathbf{v} \cdot \nabla)(\theta + \eta) = \text{Ra} w + \text{L}\nabla^2 \eta \quad (13)$$

$$(1 + \Gamma D_t)\tau = (1 + \Lambda \Gamma D_t)\mathbf{D}, \quad (14)$$

where the following dimensionless numbers have been introduced: a) buoyancy in pure fluid is characterized by the Rayleigh number:  $\text{Ra} = \alpha g \Delta T d^3 / (\kappa \nu)$ ; b) the ratio between viscous and thermal diffusivities is given by the Prandtl number  $\text{P} = \nu / \kappa$ ; c) the rotation rate is quantified by the Taylor number  $T_a = 4\varpi^2 d^4 / \nu^2$ ; the ratio between the compositional and thermal diffusivities in a binary fluid is given by the Lewis number  $\text{L} = D / \kappa$  and the separation ratio  $\psi = \beta k_T / (\alpha T_0)$  amounts for the Soret effect; d) the viscoelastic character of the liquid mixture appears in the Deborah number  $\Gamma = \lambda_1 \kappa / d^2$  and in the ratio between the stress relaxation and retardation times  $\Lambda = \lambda_2 / \lambda_1$  that varies from 0 for a Maxwell fluid to one for a Newtonian fluid [22]. The parameters  $\text{Ra}$  and  $T_a$  can be varied during an experiment by several order of magnitude, while the parameter  $\Gamma$  that relates the relaxation time to the thermal diffusion time is of order one for most viscoelastic fluids. The remaining parameters  $\text{P}$ ,  $\Lambda$ ,  $\text{L}$  and  $\psi$  depend only of the liquid mixture and can not be modified during an experiment. A

typical value for the Prandtl number is  $P \approx 10$  (dilute DNA suspension) [7]. For the case of interest in the present paper, i.e. aqueous suspensions of DNA,  $L$  is very small  $L \in [10^{-5}, 10^{-4}]$  [3] and experiments performed by Chu and collaborators [9–12] suggest that the value for the Deborah number in DNA suspensions is most likely in the range  $\Gamma \in [0.1, 2]$ . Unfortunately, at present time, there is no experimental data available neither for the separation ratio  $\psi$  nor for the ratio between the stress relaxation and retardation times  $\Lambda$ , therefore we use after looking for comparable binary mixtures a slightly negative value for  $\psi$  and parameter  $\Lambda$  in the range  $[0, 1]$ .

### 3 Linear stability analysis

The linearized version of equations (10)–(14) is easily obtained by neglecting the advective terms ( $\mathbf{v} \cdot \nabla$ ) and replacing  $D_t$  by  $\partial_t$  [15]. The centrifugal term can be included in the pressure term. This last term is then eliminated by applying twice the curl operator to the momentum equation. Using standard techniques [15], the spatial and temporal dependencies are separated using normal mode expansion,  $(\theta, \eta, w, \xi)(\mathbf{r}, t) = (\Theta, \Phi, W, Z)(z) \exp(i\mathbf{k}_\perp \cdot \mathbf{r}_\perp + st)$ , where  $\xi$  denotes the  $z$ -component of the vorticity, the subscript  $\perp$  indicates functions in the horizontal plane only,  $\mathbf{k}_\perp$  is the horizontal wavenumber of the infinitesimal perturbation (with modulus  $k$ ) and  $s = \sigma_r + i\Omega$  is the associated complex eigenvalue, with  $\sigma_r$  the linear growth factor of the perturbation and  $\Omega$  its frequency. After some straightforward algebra, equations (10)–(14) are linearized to the following coupled ordinary differential equations:

$$[D^2 - k^2 - s]\Theta + \text{Ra} W = 0 \quad (15)$$

$$[L(D^2 - k^2) - s]\Phi - s\Theta + \text{Ra} W = 0 \quad (16)$$

$$Q[D^2 - k^2 - s/(\text{PQ})]Z + \sqrt{T_a} DW = 0 \quad (17)$$

$$Q[D^4 - (2k^2 + s/(\text{PQ}))D^2 + k^2(k^2 + s/(\text{PQ}))]W - \sqrt{T_a} DZ - k^2[\psi\Phi + (1 + \psi)\Theta] = 0, \quad (18)$$

where  $Q = (1 + \Lambda\Gamma s)/(1 + \Gamma s)$  and  $D^n f = d^n f/dz^n$  is the ordinary derivative in the  $z$  direction. In the following two subsections, we analyze the results of the linear stability analysis for the two considered boundary conditions.

#### 3.1 Idealized boundary conditions

In the case of free–free boundary conditions which would correspond to a fluid with no viscosity at the boundary (slippery walls), it is possible to solve analytically the system of equations (15)–(18). This is a standard eigenvalue–eigenfunction problem. The parameter Rayleigh number ( $\text{Ra}$ ) is taken as the eigenvalue and it is expressed as a function of  $s, q$  and the remaining parameters of the problem. An explicit expression of the Rayleigh number is given:

$$\text{Ra} = \frac{(q_n^2 + s)(Lq_n^2 + s)}{k^2(1 + \psi)[s + q_n^2(L + \gamma)]} \left\{ \frac{Pn^2\pi^2 T_a(1 + s\Gamma)}{s(1 + s\Gamma) + Pq_n^2(1 + s\Gamma\Lambda)} + \frac{q_n^2[s(1 + s\Gamma) + Pq_n^2(1 + s\Gamma\Lambda)]}{P(1 + s\Gamma)} \right\}, \quad (19)$$

where  $q_n^2 = k^2 + (n\pi)^2$  is a global wavenumber that takes into account the horizontal wavenumber and the order of the vertical eigenfunction ( $n$  is a positive integer), the maximum instability i.e. minimum eigenvalue is obtained for  $n = 1$  (fundamental mode). Another new parameter  $\gamma = \psi/(1 + \psi)$  is introduced to simplify the above mathematical expression. Let us now discuss some limiting cases of the expression equation (19). If one sets the rotation rate to zero, one gets the following expression:

$$\lim_{T_a \rightarrow 0} \text{Ra} = \frac{q_n^2(q_n^2 + s)(Lq_n^2 + s)[s(1 + s\Gamma) + Pq_n^2(1 + s\Gamma\Lambda)]}{Pk^2(1 + \psi)(1 + s\Gamma)[s + q_n^2(L + \gamma)]}, \quad (20)$$

this case has been studied thoroughly by Martínez–Mardones et al. [14]. Secondly, considering the smallness of  $L$  in molecular mixtures  $L \approx 10^{-2}$  and in polymeric suspensions

$L \in [10^{-4}, 10^{-5}]$  [23,24], one takes the limit  $L = 0$  in equation (19) to obtain the following expression:

$$\lim_{L \rightarrow 0} \text{Ra} = \frac{s(q_n^2 + s)}{k^2(1 + \psi)(s + q_n^2\gamma)} \left\{ \frac{Pn^2\pi^2T_a(1 + s\Gamma)}{s(1 + s\Gamma) + Pq_n^2(1 + s\Gamma\Lambda)} + \frac{q_n^2[s(1 + s\Gamma) + Pq_n^2(1 + s\Gamma\Lambda)]}{P(1 + s\Gamma)} \right\}. \quad (21)$$

Thirdly, taking the limit of a very viscous liquid  $P \rightarrow \infty$ , one obtains from equation (19) the following expression:

$$\lim_{P \rightarrow \infty} \text{Ra} = \frac{(q_n^2 + s)(Lq_n^2 + s)}{k^2q_n^2(1 + \psi)[s + q_n^2(L + \gamma)]} \left\{ \frac{n^2\pi^2T_a(1 + s\Gamma)}{1 + s\Gamma\Lambda} + \frac{q_n^6(1 + s\Gamma\Lambda)}{1 + s\Gamma} \right\}. \quad (22)$$

Fourthly, when both  $L$  and  $\psi$  vanish, one recovers the pure rotating viscoelastic fluid and equation (19) takes the following form:

$$\lim_{L; \psi \rightarrow 0} \text{Ra} = \frac{q_n^2 + s}{k^2} \left\{ \frac{Pn^2\pi^2T_a(1 + s\Gamma)}{s(1 + s\Gamma) + Pq_n^2(1 + s\Gamma\Lambda)} + \frac{q_n^2[s(1 + s\Gamma) + Pq_n^2(1 + s\Gamma\Lambda)]}{P(1 + s\Gamma)} \right\}, \quad (23)$$

note that in this last case the fraction in front of the curly brackets is considerably reduced. In particular, both numerator and denominator have lost one power of  $s$ . Physically, it is due to the fact that taking a pure fluid rather than a binary mixture one reduces the degree of freedom for the appearance of instabilities. This last situation described by equation (23) has been studied previously in [20,25]. Fifthly, when  $\Gamma = 0$  in equation (19), one eliminates the viscoelastic character of the liquid and one gets the following expression for the Rayleigh number for rotating binary mixtures:

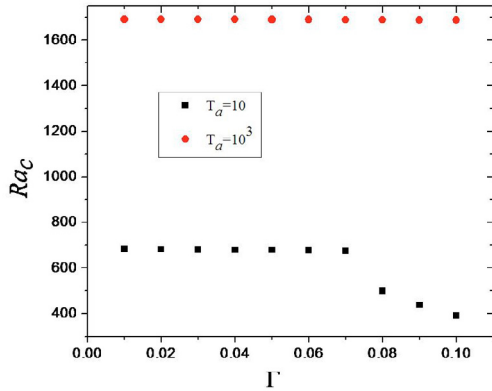
$$\lim_{\Gamma \rightarrow 0} \text{Ra} = \frac{(q_n^2 + s)(Lq_n^2 + s)}{k^2(1 + \psi)[s + q_n^2(L + \gamma)]} \left\{ \frac{Pn^2\pi^2T_a}{s + Pq_n^2} + \frac{q_n^2(s + Pq_n^2)}{P} \right\}, \quad (24)$$

note that in the equation (24) the degree of the numerator and denominator have been reduced by a factor one in term of  $s$ , the complex growth factor. Again, this is due to the fact that removing the viscoelastic character of the liquid, one loses a mechanism for instability to set up. These analytical considerations are very interesting because, one sees that the above limit  $\Gamma \rightarrow 0$  is singular and numerically it can lead to artefacts if it is not appropriately dealt with. The study of equation (24) has been previously considered by Bhattacharjee [17,18] and Kumar [19].

Finally, let us consider the case of interest of this paper i.e. a viscoelastic binary liquid mixture heated from below. As equation (19) is not amenable of analytical treatments, we can proceed with further simplifications. As we have already commented before, these simplifications must not alter the generality of the analysis, i.e. one can not reduce the degree of the numerator or denominator in term of  $s$ , the linear complex growth factor. Two regular (opposed to singular) limiting cases are however possible in equation (19), one is obtained by taking the limit of very viscous mixtures  $P \rightarrow \infty$  and the second simplification is obtained by considering a Maxwellian mixture  $\Lambda = 0$ , one gets for the Rayleigh number:

$$\text{Ra}(s) = \frac{(q_n^2 + s)(Lq_n^2 + s)}{k^2q_n^2(1 + \psi)[s + q_n^2(L + \gamma)]} \left\{ n^2\pi^2T_a(1 + s\Gamma) + \frac{q_n^6}{1 + s\Gamma} \right\}, \quad (25)$$

this last expression is an analytical function of the complex variable  $s$  and one can use the standard tools of complex analysis to study this function [26]. In particular, the threshold for linear instability is obtained by setting  $\sigma_r = 0$  and  $s = i\Omega$  in the complex expression equation (25). Due to the fact that the Rayleigh number is a physical parameter, the imaginary part of equation (25) must vanish and the real part gives the marginal stability curve  $\text{Ra} = \text{Ra}(k, \Omega)$ . This condition, together with the condition of an extremum of the function at the critical point  $\partial_k(\text{Re}[\text{Ra}])|_{k=k_c} = 0$  allow to determine the critical wavenumber  $k_c$  and the



**Fig. 1.** Free–Free boundary conditions: Critical Rayleigh number as a function of the Deborah number for the oscillatory instability of a rotating Maxwellian mixture. Parameters values are  $L = 10^{-4}$ ;  $\psi = -10^{-4}$  and two rotation rates  $T_a = 10$  (black squares) and  $T_a = 10^3$  (red dots).

frequency  $\Omega_c$  of the linearly most dangerous perturbation and its corresponding critical Rayleigh number  $Ra_c$ .

Figure 1 displays the variation of the critical Rayleigh number as a function of the Deborah number  $\Gamma$  for a rotating Maxwellian mixture ( $\Lambda = 0$ ) and for two different values of the rotation rate  $T_a = 10$  and  $T_a = 10^3$  (the angular velocities corresponding to these values of the Taylor number, for a depth layer of one centimetre, correspond to approximately  $1.6 \times 10^{-2}$  (rad/s) and  $1.6 \times 10^{-1}$  (rad/s), respectively), the parameters related to the binary character of the mixture have been set in order to describe as closely as possible DNA suspensions, the Lewis number is set to  $L = 10^{-4}$  and the separation ratio is set to a small negative value  $\psi = -10^{-4}$ . For low values of the Deborah number  $\Gamma$  the critical Rayleigh number  $Ra_c$  remains nearly constant while it substantially decrease at larger value of the Deborah number. Also, the rotation rate modify the value at which the transition is observed. In particular, the larger is the Taylor number, the larger is the Deborah number at which the transition occurs.

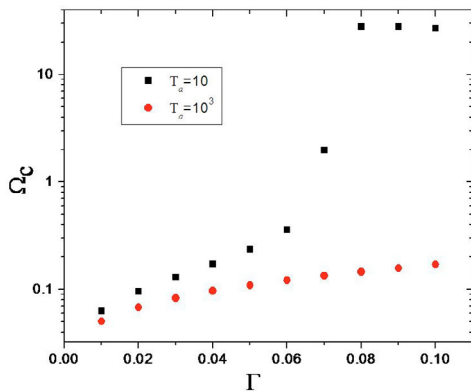
At this point, it may be useful to remind the physical meaning of the Deborah number [27]. The Deborah number is a dimensionless number, used in rheology to characterize how “fluid” a material is. Even some apparent solids “flow” if they are observed long enough; the origin of the name is the line “The mountains flowed before the Lord” in a song by prophetess Deborah recorded in the Bible (Judges 5:5). Formally, the Deborah number is defined as the ratio of a relaxation time, characterizing the intrinsic fluidity of a material, and the characteristic time scale of an experiment (or a computer simulation) probing the response of the material. The smaller the Deborah number, the more fluid the material appears.

Therefore, the results for the thermal convection in a rotating Maxwellian mixture indicate that when considering a stiffer fluid (due to its viscoelastic character) its thermal convective threshold is lowered. This is not an intuitive result. Another result which is common with the Newtonian fluids is that the rotation has a stabilizing effect on the layer, the larger is the Taylor number, the larger is the instability threshold  $Ra_c$ .

Figure 2 displays the frequency corresponding to the most dangerous perturbation at threshold ( $\Omega_c$ ) as a function of the Deborah number  $\Gamma$ . This plot must be put in direct correspondence to the figure 1, in particular, the mild inflexion in the curve of  $Ra_c$  versus  $\Gamma$  shown in figure 1 corresponds to a strong discontinuity in the curve of ( $\Omega_c$ ) versus  $\Gamma$  as shown in figure 2. Note that the scale in figure 2 is logarithmic. After the discontinuity, the frequency remains quasi constant for increasing values of  $\Gamma$ .

### 3.2 Realistic boundary conditions

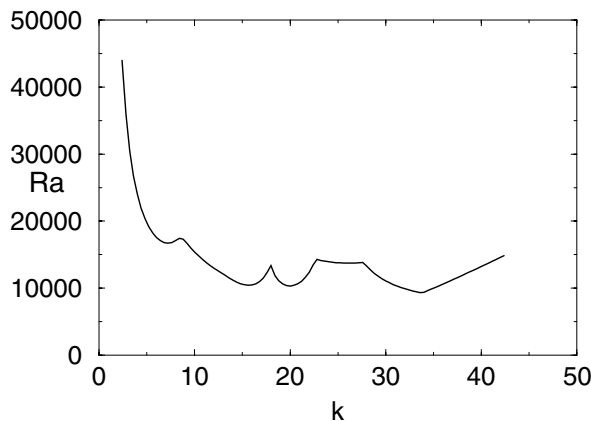
The use of free–free boundary conditions ( $W = D^2W = 0$ ) at the two horizontal boundaries is a useful mathematical simplification but it is not physically sound. The correct boundary conditions for a viscous or viscoelastic fluid is to impose ( $W = DW = 0$ ) at the two horizontal boundaries. In order to solve the set of ordinary differential equations (15)–(18), we



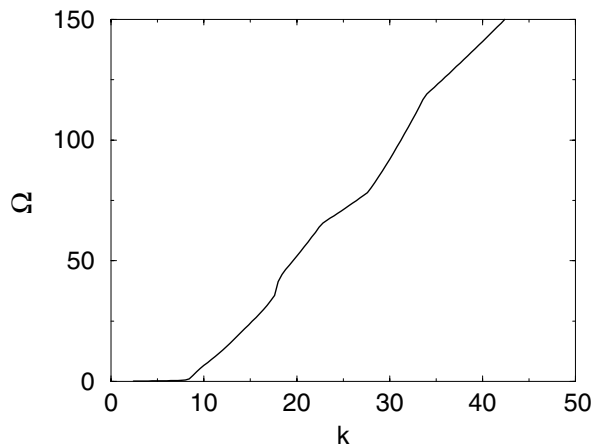
**Fig. 2.** Free-Free boundary conditions: Frequency of the perturbation at threshold as a function of the Deborah number for the oscillatory instability of a rotating Maxwellian mixture. Parameters values are the same as in figure 1.

have used a spectral method. Spectral methods ensure an exponential convergence to the solution and are the best available numerical techniques for solving simple eigenvalue-eigenfunction problems. Here, we have followed the technique of collocation points on a Chebyshev grid as described by Threfethen [28]. We have determined that for the resolution of equations (15)–(18) 14 collocation points in the vertical direction are enough to ensure spectral convergence. The collocation points (Gauss-Lobato) are located at height:  $z_j = \cos(j\pi/N)$  where index  $j$  runs from  $j = 0$  to  $j = N$  (here  $N = 14$ ) [28]. The equations and the boundary conditions are expressed at the collocation points. By doing so, the set of differential equations (15)–(18) is transformed into a set of algebraic equations. The eigenfunctions  $(\Theta(z), \Phi(z), W(z), Z(z))$  are transformed into eigenvectors defined at the collocation points. The Rayleigh number is again used as the eigenvalue of the problem. After this stage of discretization, one is left with a classical eigenvalue-eigenvector problem that can be easily solved by using a Matlab routine.

Since the Rayleigh number is a physical quantity which is real, one has to select  $\Omega$  in order to get a vanishing imaginary part for the Rayleigh number  $Ra$ . As it has been shown in equation (25) for the free-free case and due to the fact that the expression for the Rayleigh number is a ratio of polynomials in  $s$ , one can get multiple solutions (branches in the complex plane) i.e. various values of  $\Omega$  that make the imaginary part of  $Ra$  to vanish. In case of multiple solutions, we have always selected the one that corresponds to the lowest value of the Rayleigh number (the most unstable one). By doing so, one is left with a triplet  $(Ra, k, \Omega)$  that define a marginal stability condition. This procedure is repeated for several values of the horizontal wavenumber  $k$ . These points serve to draw the marginal stability curve  $Ra$  versus  $k$ . The minimum of this curve gives the  $Ra_c$  and  $k_c$ , and the corresponding value for the critical frequency  $\Omega_c$ .



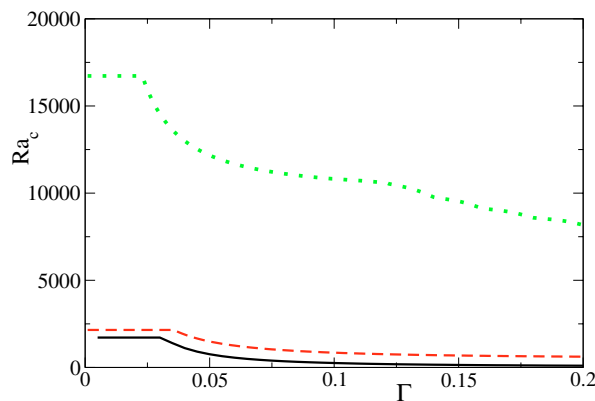
**Fig. 3.** Rigid-Rigid boundary conditions: Stability curve, the Rayleigh number as function of the wavenumber  $k$ , showing multistability. Parameters values are  $L = 10^{-4}$ ;  $\psi = -10^{-4}$ ;  $\Gamma = 0.155$  and rotation rate  $T_a = 10^5$ .



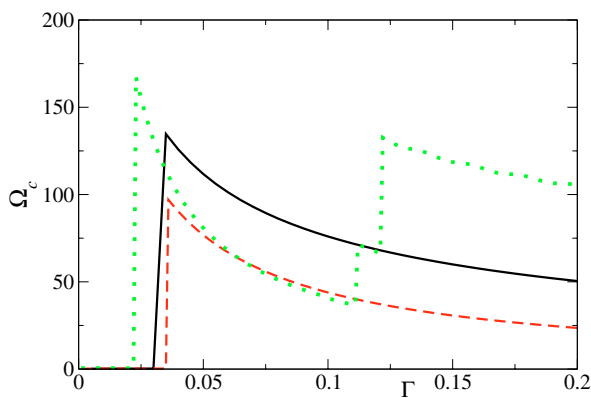
**Fig. 4.** Rigid–Rigid boundary conditions: Frequency versus wavenumber, all the minima of figure 3 are oscillatory (non vanishing frequency). Parameters values are the same as in figure 3.

Figure 3 shows the stability curve for  $T_a = 10^5$  and  $\Gamma = 0.155$ . The results of the stability analysis in the case of rigid–rigid conditions are displayed in figures 5 and 6.

Using the same criterion as for figures 1 and 2, one depicts the variations of  $Ra_c$  and  $\Omega_c$  as function of the Deborah number  $\Gamma$  for a rotating Maxwellian mixture. One observes that as usual in thermal convection, the rotation has a stabilizing effect i.e. the critical Rayleigh number is increased when the Taylor number is increased (see figure 5). In addition, the influence of the Deborah number is more pronounced in the rigid–rigid case than in the free–free case and especially for high rotation rate, this is presumably due to the stronger coupling of the vorticity



**Fig. 5.** Rigid–Rigid boundary conditions: Critical Rayleigh number as a function of the Deborah number ( $\Gamma$ ). Parameters values are  $L = 10^{-4}$ ;  $\psi = -10^{-4}$ ; and rotation rates  $T_a = 10$  (black solid line),  $T_a = 10^3$  (red dashed line),  $T_a = 10^5$  (green dotted line).



**Fig. 6.** Rigid–Rigid boundary conditions: Critical frequency as a function of the Deborah number ( $\Gamma$ ) showing that for large Taylor discontinuities in the critical frequency are frequent. Parameters values are the same as in figure 5.



field and the viscoelastic character of the fluid when realistic rigid–rigid boundary conditions are used. The most striking difference between the two cases is found by comparing the critical frequencies (see figures 2 and 6), in the rigid–rigid case, the critical frequency is of order one hundred and experiences many discontinuities due to competing instability mechanisms when the Deborah number is increased while for the free–free case, the frequencies are order of magnitude smaller and undergo discontinuities for larger Deborah number. In the present problem, one has four potential sources of instability (thermal, binary, rotation and viscoelastic character) which lead to several possible exchange of stability or discontinuity in the critical curves as it is clearly observed in figure 3.

## 4 Conclusions

In the present work, Rayleigh–Bénard convection in a Maxwellian binary mixture under rotation has been studied. We have determined the stability thresholds for oscillatory convection for both free–free and rigid–rigid boundary conditions. Due to the mixing of several motors for instability i.e. buoyancy, compositional gradients, rotation and viscoelastic character, the stability curves exhibit several branches. These different instability mechanisms are competing when the Deborah number is varied. One observes that the critical frequency is a discontinuous function (see figure 6) when the Deborah number is increased. In planning new experiments with binary mixtures of viscoelastic liquid (in particular DNA suspensions), one has to be very careful because one sees that minute changes in parameters may induce large variations in thresholds and frequencies due to the complexity of the problem. These recommendations apply also for the design of new PCR techniques based on Rayleigh–Bénard convection cells of DNA suspensions.

We emotionally dedicate this article to the memory of Prof. Carlos Pérez-García (Q.E.P.D.) for his motivations and fruitful interchange of ideas about hydrodynamics instabilities.

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