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Dynamics of a rotating particle under a time-dependent potential: exact quantum solution from the classical action

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Abstract
The classical action for a three-dimensional rotating particle with time-dependent angular velocity and that submitted to a general linear and quadratic time-dependent potential is explicitly calculated and, via path integral approach, the quantum propagator is obtained. For some special values of the coefficients, it is also shown that the quantum propagator in the rotating frame contains a phase factor which gives rise to a quantum interference phenomenon.

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1. Introduction

Quantum systems with an explicit time-dependent Hamiltonian have attracted much attention over the past decades. In addition to their intrinsic mathematical interest, these systems connect to several important physical problems such as quantum optics [1], Paul trap [2], Berry phase [3] and magnetic field lens [4, 5]. Unfortunately, not all time-dependent systems can be solved analytically. In fact, there are only a few with already known solutions, most remarkably the time-dependent quadratic Hamiltonian, for which analytical results exist in the one-dimensional case [6–10]. Interestingly, the problem of an electron submitted to a time-dependent magnetic field [6, 11, 12], or in general, a particle under a kind of Coriolis force [13], can be mapped to the quadratic case. Thus, to have a general, three-dimensional solution of the problem appears highly desirable.

An important example of a time-dependent system is the one under rotation. The physical effects produced by a rotating reference frame are studied in many areas of physics, such as classical dynamics [13], fluid dynamics [14], electromagnetic theory [15], quantum mechanics [16–19], and quantum field theory [20–22]; furthermore, it is an experimental fact that the rotation produces quantum interference [23–25]. In addition, when the angular velocity is not constant new effects arise, and in this case the adiabatic and non-adiabatic Berry topological phases have been calculated [3, 26–28].

In the present work, we analyze in detail the non-relativistic classical as well as the quantum behaviors of a particle seen from a frame which rotates with a time-dependent angular velocity ω and submitted to time-dependent potentials which are linear and quadratic in position. In order to obtain a solution to the time-dependent Schrödinger equation we adopt the following approach. First, we solve the classical equation of motion of the system, and evaluate the classical action. Then, we attack the quantum problem by expressing the propagator in terms of a path integral, exploiting the fact that the Lagrangian is a quadratic function of the position and the velocity. Thus, the propagator becomes a product of a term that contains the classical action times a functional integral which can be evaluated exactly. In this way, we obtain an explicit, closed solution which allows us to consider different cases of interest. The paper is arranged as follows: section 2 is devoted to the classical dynamics analysis, section 3 presents the quantum dynamics problem and conclusions are drawn in section 4.

2. Classical dynamics

Let us consider a particle of mass m submitted to a potential $U(\boldsymbol{r}, t)$. Its Lagrangian L with respect to an inertial reference frame $Oxyz$ would be

$$L(\boldsymbol{r}, \dot{\boldsymbol{r}}, t) = \frac{m\dot{\boldsymbol{r}}^2}{2} - U(\boldsymbol{r}, t). \tag{1}$$

Let $Ox'y'z'$ be another reference frame, which rotates with respect to $Oxyz$ in a fixed axis with a time-dependent angular velocity $\boldsymbol{\omega}(t)$. Then, the Lagrangian in $Ox'y'z'$ would be given by

$$\mathcal{L}(\boldsymbol{r}', \dot{\boldsymbol{r}}', t) = \frac{m}{2}\dot{\boldsymbol{r}}'^2 + \mathrm{m}\dot{\boldsymbol{r}}' \cdot (\boldsymbol{\omega} \times \boldsymbol{r}') + \frac{m}{2}(\boldsymbol{\omega} \times \boldsymbol{r}')^2 - \mathcal{U}(\boldsymbol{r}', t). \tag{2}$$

Notice that if we use the potentials

$$A(\boldsymbol{r}', t) = \frac{m}{q}\boldsymbol{\omega} \times \boldsymbol{r}' \text{ and } \phi(\boldsymbol{r}', t) = \frac{m}{2q}(\boldsymbol{\omega} \times \boldsymbol{r}')^2,$$

the Lagrangian of equation (2) can be put in the form of the Lagrangian of a particle of charge q moving in the electromagnetic field $\boldsymbol{E} = -\nabla'\phi - \partial \boldsymbol{A}/\partial t$, $\boldsymbol{B} = \nabla' \times \boldsymbol{A}$.

The classical action $\mathcal{S}_{\mathrm{cl}}$ is given by

$$\mathcal{S}_{\mathrm{cl}}(\boldsymbol{r}'_f, t_f; \boldsymbol{r}'_i, t_i) = \int_{t_i}^{t_f} \mathcal{L}(\boldsymbol{r}', \dot{\boldsymbol{r}}', t)\, \mathrm{d}t, \tag{3}$$

where $\boldsymbol{r}'_i = \boldsymbol{r}'(t_i)$ and $\boldsymbol{r}'_f = \boldsymbol{r}'(t_f)$. We denote the functions in the rotating frame $Ox'y'z'$ with calligraphic letters.

Let us consider hereafter a particular time-dependent potential which contains only a linear and a quadratic term in the position, $U(\boldsymbol{r}, t) = -m\boldsymbol{\alpha}(t) \cdot \boldsymbol{r} + (1/2)m\beta^2(t)\boldsymbol{r}^2$. In the reference frame $Ox'y'z'$, the Lagrangian (equation (2)) becomes

$$\begin{aligned}\mathcal{L}(\boldsymbol{r}', \dot{\boldsymbol{r}}', t) = {}& \frac{m}{2}\dot{\boldsymbol{r}}'^2 + m\dot{\boldsymbol{r}}' \cdot (\boldsymbol{\omega} \times \boldsymbol{r}') + \frac{m}{2}(\boldsymbol{\omega} \times \boldsymbol{r}')^2 \\ & + m\boldsymbol{\alpha}'(t) \cdot \boldsymbol{r}' - \tfrac{1}{2}m\beta^2(t)\boldsymbol{r}'^2.\end{aligned} \tag{4}$$

and the equation of motion is

$$\ddot{\boldsymbol{r}}' - \boldsymbol{r}' \times \dot{\boldsymbol{\omega}} - 2\dot{\boldsymbol{r}}' \times \boldsymbol{\omega} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r}') + \beta^2(t)\boldsymbol{r}' = \boldsymbol{\alpha}'(t)\,. \tag{5}$$

In order to solve this equation, we put $\boldsymbol{\omega}(t) = \omega(t)\hat{z}$, obtaining

$$\ddot{x}' + \left(-\omega^2 + \beta^2\right) x' - y'\dot{\omega} - 2\dot{y}'\omega = \alpha'_x, \tag{6}$$

$$\ddot{y}' + \left(-\omega^2 + \beta^2\right) y' + x'\dot{\omega} + 2\dot{x}'\omega = \alpha'_y, \tag{7}$$

$$\ddot{z}' + \beta^2 z' = \alpha'_z. \tag{8}$$

By means of the transformation $\eta = x' + \mathrm{i}y'$, equations (6) and (7) become

$$\ddot{\eta} + (-\omega^2 + \beta^2)\eta + \mathrm{i}\eta\dot{\omega} + 2\mathrm{i}\dot{\eta}\omega = \alpha'_\eta\,, \tag{9}$$

where $\alpha'_\eta = \alpha'_x + \mathrm{i}\alpha'_y$. To solve this equation, first we look for solution of the homogeneous part. Performing the change of variables [29] $\eta = \rho \exp(-\mathrm{i}\theta(t, t_i))$, where

$$\theta(t, t_i) = \int_{t_i}^{t} \omega(\tilde{t})\, \mathrm{d}\tilde{t}. \tag{10}$$

and replacing η in the homogeneous part of equation (9), we obtain the well-known Hill equation [30]

$$\ddot{\rho} + \beta^2(t)\rho = 0. \tag{11}$$

Note that, since β^2 is always a positive function, all the non-trivial solutions of equation (11) have an oscillatory nature, and therefore there exists an infinite sequence of values of t_q for which $\rho(t_q) = 0$. The solutions of this equation allows us to write a formal solution, $\boldsymbol{r}'_h(t)$, of the homogeneous part of equation (5) as

$$\boldsymbol{r}'_h(t) = \mathsf{R}(\theta(t, t_i))(\boldsymbol{C}_1 s_1(t, t_i) + \boldsymbol{C}_2 s_2(t, t_i)), \tag{12}$$

where

$$\mathsf{R}(\theta) = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{13}$$

and s_1, s_2 is a pair of real solutions of equation (11) with initial conditions $s_1(t_i, t_i) = 0$, $\dot{s}_1(t_i, t_i) = 1$ and $s_2(t_i, t_i) = 1$, $\dot{s}_2(t_i, t_i) = 0$. Notice that $s_1(t, t_i)$ has dimension of time, $s_2(t, t_i)$ is adimensional and the wronskian of s_1 and s_2 is one. $\boldsymbol{C}_1$ and $\boldsymbol{C}_2$are two constant vectors determined by the initial conditions, and their values are $\boldsymbol{C}_1 = \dot{\boldsymbol{r}}'_i + \boldsymbol{\omega}_i \times \boldsymbol{r}'_i$ and $\boldsymbol{C}_2 = \boldsymbol{r}'_i$, where $\boldsymbol{\omega}_i = \boldsymbol{\omega}(t_i)$. Thus, the homogeneous solution (equation (12)) can be written as

$$\boldsymbol{r}'_h(t) = \mathsf{R}\big(\theta(t, t_i)\big)\Big((\dot{\boldsymbol{r}}'_i + \boldsymbol{\omega}_i \times \boldsymbol{r}'_i)s_1(t, t_i) + \boldsymbol{r}'_i s_2(t, t_i)\Big). \tag{14}$$

The non-homogeneous equation can be solved by the method of variation of parameters [31], giving

$$\boldsymbol{r}'(t) = \boldsymbol{r}'_h(t) + \int_{t_i}^{t} G(t, \tilde{t})\mathsf{R}\big(\theta(t, t_i) - \theta(\tilde{t}, t_i)\big)\boldsymbol{\alpha}'(\tilde{t})\, \mathrm{d}\tilde{t}\,, \tag{15}$$

where the propagator $G(t, \tilde{t})$ satisfies

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}t^2} + \beta^2\right) G(t, \tilde{t}) = \delta(t - \tilde{t}) \tag{16}$$

with the same initial condition used for the homogeneous part. The formal solution of this equation is

$$G(t, \tilde{t}) = s_1(t, t_i)s_2(\tilde{t}, t_i) - s_1(\tilde{t}, t_i)s_2(t, t_i)\,. \tag{17}$$

Of course, we recover the solution of the problem in the frame $Oxyz$ by making $\omega(t) \equiv 0$ in equation (14),

$$\begin{aligned}\boldsymbol{r}(t) = {}& \dot{\boldsymbol{r}}_i s_1(t, t_i) + \boldsymbol{r}_i s_2(t, t_i) + s_1(t, t_i)\int_{t_i}^{t} s_2(\tilde{t}, t_i)\boldsymbol{\alpha}(\tilde{t})\, \mathrm{d}\tilde{t} \\ & - s_2(t, t_i)\int_{t_i}^{t} s_1(\tilde{t}, t_i)\boldsymbol{\alpha}(\tilde{t})\, \mathrm{d}\tilde{t}\,.\end{aligned} \tag{18}$$

In the simple case of a free particle in a rotating frame, equation (14) becomes

$$\boldsymbol{r}_0'(t)=\mathsf{R}\big(\theta(t,t_i)\big)\Big(\big(\dot{\boldsymbol{r}}_i'+\boldsymbol{\omega}_i\times\boldsymbol{r}_i'\big)(t-t_i)+\boldsymbol{r}_i'\Big). \tag{19}$$

Now, we study the classical action which will be used to obtain the quantum dynamics. The classical action $\mathcal{S}_{\rm cl}$ can be calculated by using the equation (15), obtaining

$$\mathcal{S}_{\rm cl}(\boldsymbol{r}_f',t_f;\boldsymbol{r}_i',t_i)=\frac{m}{2}\boldsymbol{r}'(t)\cdot\dot{\boldsymbol{r}}'(t)\Big|_{t_i}^{t_f}+\frac{m}{2}\int_{t_i}^{t_f}\boldsymbol{r}'(t)\cdot\boldsymbol{\alpha}'(t)\,\mathrm{d}t. \tag{20}$$

After several manipulations, we obtain

$$\begin{aligned}
\mathcal{S}_{\rm cl}(\boldsymbol{r}_f',t_f;\boldsymbol{r}_i',t_i)&=\frac{m}{2s_1(t_f,t_i)}\\
&\times(\dot{s}_1(t_f,t_i)\boldsymbol{r}_f'^2+s_2(t_f,t_i)\boldsymbol{r}_i'^2-2\boldsymbol{r}_f'\cdot\mathsf{R}(\theta(t_f,t_i))\boldsymbol{r}_i')\\
&+2\dot{s}_1(t_f,t_i)\mathtt{I}_2(\theta(t,t_i)-\theta(t_f,t_i),\boldsymbol{r}_f')\\
&-2\dot{s}_2(t_f,t_i)\mathtt{I}_1(\theta(t,t_i)-\theta(t_f,t_i),\boldsymbol{r}_f')\\
&+\dot{s}_1(t_f,t_i)s_1(t_f,t_i)\mathtt{A}_{2,2}(\theta(\bar{t},t_i)-\theta(\tilde{t},t_i))\\
&-\dot{s}_2(t_f,t_i)s_1(t_f,t_i)\mathtt{A}_{2,1}(\theta(\bar{t},t_i)-\theta(\tilde{t},t_i))\\
&-\dot{s}_1(t_f,t_i)s_2(t_f,t_i)\mathtt{A}_{1,2}(\theta(\bar{t},t_i)-\theta(\tilde{t},t_i))\\
&+\dot{s}_2(t_f,t_i)s_2(t_f,t_i)\mathtt{A}_{1,1}(\theta(\bar{t},t_i)-\theta(\tilde{t},t_i))\\
&+\frac{m}{2}\int_{t_i}^{t_f}\int_{t_i}^{\bar{t}}\boldsymbol{\alpha}'(\bar{t})\cdot\mathsf{R}(\theta(\bar{t},t_i)-\theta(\tilde{t},t_i))\boldsymbol{\alpha}'(\tilde{t})\\
&\times(s_1(\bar{t},t_i)s_2(\tilde{t},t_i)-s_2(\tilde{t},t_i)s_1(\bar{t},t_i))\,\mathrm{d}\tilde{t}\,\mathrm{d}\bar{t},
\end{aligned} \tag{21}$$

where

$$\mathtt{I}_\mu(q,\boldsymbol{c})=\frac{m}{2}\int_{t_a}^{t_b}\boldsymbol{\alpha}'(\bar{t})\cdot\mathsf{R}(q)\boldsymbol{c}(\bar{t})s_\mu(t_i,\bar{t})\,\mathrm{d}\bar{t}, \tag{22}$$

$$\mathtt{A}_{\mu,\nu}(q)=\frac{m}{2}\int_{t_a}^{t_b}\int_{t_a}^{t_b}\boldsymbol{\alpha}'(\bar{t})\cdot\mathsf{R}(q)\boldsymbol{\alpha}'(\tilde{t})s_\mu(\bar{t})s_\nu(\tilde{t})\,\mathrm{d}\tilde{t}\,\mathrm{d}\bar{t} \tag{23}$$

and $\mu=1,2$, $\nu=1,2$. This result, to the best of our knowledge not reported in the literature so far, contains in one formula the combined effects that a time-dependent rotating frame and a linear and quadratic time-dependent potential produce on a particle. It is interesting to evaluate this classical action in some particular cases.

For the motion of the particle in a non-rotating frame, i.e. $\boldsymbol{\omega}=0$, and only submitted to the linear and quadratic time-dependent potential, equation (21) takes a simpler form, since the angle θ vanishes and therefore all the rotation matrices become the identity matrix. In particular, if $\boldsymbol{\alpha}=\boldsymbol{0}$ then the particle is only submitted to the quadratic potential, and we obtain

$$\begin{aligned}\mathcal{S}_{\rm cl}(\boldsymbol{r}_f,t_f;\boldsymbol{r}_i,t_i)&=\frac{m}{2s_1(t_f,t_i)}(\dot{s}_1(t_f,t_i)\boldsymbol{r}_f^2\\
&+s_2(t_f,t_i)\boldsymbol{r}_i^2-2\boldsymbol{r}_f\cdot\boldsymbol{r}_i).\end{aligned} \tag{24}$$

Consider now the classical action for the particle in the rotating reference frame, but $\boldsymbol{\alpha}'=\boldsymbol{0}$ and $\beta=0$, that is, not submitted to any external potential. Then,

$$\mathcal{S}_{\rm cl}(\boldsymbol{r}_f',t_f';\boldsymbol{r}_i',t_i')=\frac{m}{2(t_f-t_i)}\left(\boldsymbol{r}_f'^2+\boldsymbol{r}_i'^2-2\boldsymbol{r}_f'\cdot\mathsf{R}(\theta(t_f,t_i))\boldsymbol{r}_i'\right). \tag{25}$$

In this case, of course, when $\boldsymbol{\omega}=0$ we recover the well-known result [32] of the classical action for a free particle

$$\mathcal{S}_{\rm cl}(\boldsymbol{r}_f,t_f;\boldsymbol{r}_i,t_i)=\frac{m}{2(t_f-t_i)}(\boldsymbol{r}_f-\boldsymbol{r}_i)^2. \tag{26}$$

Notice that when the potential is just quadratic, $\boldsymbol{\alpha}=\boldsymbol{0}$, equation (21) is reduced to

$$\begin{aligned}\mathcal{S}_{\rm cl}(\boldsymbol{r}_f',t_f',\boldsymbol{r}_i',t_i')&=\frac{m}{2s_1(t_f,t_i)}\left(\dot{s}_1(t_f,t_i)\boldsymbol{r}_f'^2\right.\\
&\left.+s_2(t_f,t_i)\boldsymbol{r}_i'^2-2\boldsymbol{r}_f'\cdot\mathsf{R}(\theta(t_f,t_i))\boldsymbol{r}_i'\right).\end{aligned} \tag{27}$$

Interestingly, equation (27) has the functional structure similar to the one for a two-dimensional electron gas in a time-dependent magnetic field [11], it being the major difference in the rotation matrix.

Let us now consider the action difference $\Delta\mathcal{S}_{\rm cl}$ in two particular cases. First, consider the difference in the action of a particle in a rotating frame with respect to the action of a free particle. We obtain

$$\Delta\mathcal{S}_{\rm cl}=\frac{m}{(t_f-t_i)}(\boldsymbol{r}_f'\cdot(\mathsf{I}-\mathsf{R}(\theta(t_f,t_i)))\boldsymbol{r}_i'). \tag{28}$$

Similarly, in the case of the action of a particle in a rotating frame submitted to a quadratic time-dependent potential, we obtain

$$\Delta\mathcal{S}_{\rm cl}=\frac{m}{s_1(t_f,t_i)}(\boldsymbol{r}_f'\cdot(\mathsf{I}-\mathsf{R}(\theta(t_f,t_i)))\boldsymbol{r}_i'). \tag{29}$$

Notice that in order to have no difference in the action, i.e. $\Delta\mathcal{S}_{\rm cl}=0$, it is required that $\theta(t_f,t_i)=2n\pi$ with $n\in\mathbb{N}$. These last two equations are relevant for quantum interference phenomena, as we will show in section 3.

3. Quantum dynamics

Now we turn our attention to the quantum mechanical behavior of the particle. Because the Hamiltonian is time-dependent, the energy is not conserved and, consequently, there is no energy spectrum. Thus, we must tackle this problem by solving the time-dependent Schrödinger equation. For a particle in a rotating reference frame and submitted to a linear and quadratic time-dependent potential, reads as

$$\begin{aligned}&\left(-\frac{\hbar^2}{2m}\nabla'^2+\boldsymbol{\omega}\cdot\boldsymbol{L}'-m\boldsymbol{\alpha}'\cdot\boldsymbol{r}'+\tfrac{1}{2}m\beta^2\boldsymbol{r}'^2\right.\\
&\left.-\frac{m}{2}(\boldsymbol{\omega}\times\boldsymbol{r}')^2-\mathrm{i}\hbar\frac{\partial}{\partial t}\right)\psi(\boldsymbol{r}',t)=0,\end{aligned} \tag{30}$$

where $\boldsymbol{L}'$ is the angular momentum operator and the operator $\boldsymbol{\omega}\cdot\boldsymbol{L}'$ acts on a function f according to $\boldsymbol{\omega}\cdot\boldsymbol{L}'f=\mathrm{i}\hbar(\boldsymbol{\omega}\times\boldsymbol{r}')\cdot\nabla'f$. A useful method to solve this equation is by means of the propagator method. Given an initial state $\psi_0(\boldsymbol{r}_i)$ in the instant t_i, its temporal evolution to the instant t_f is given by

$$\psi(\boldsymbol{r}_f',t_f)=\int\mathrm{d}\boldsymbol{r}_i'\,G(\boldsymbol{r}_f',t_f;\boldsymbol{r}_i',t_i)\psi_0(\boldsymbol{r}_i'), \tag{31}$$

where $G(\boldsymbol{r}'_f, t_f; \boldsymbol{r}'_i, t_i)$ is the propagator, which satisfies the equation

$$\left(-\frac{\hbar^2}{2m}\nabla'^2 + \boldsymbol{\omega}\cdot\boldsymbol{L}' - m\boldsymbol{\alpha}'\cdot\boldsymbol{r}' + \tfrac{1}{2}m\beta^2\boldsymbol{r}'^2 - \tfrac{m}{2}(\boldsymbol{\omega}\times\boldsymbol{r}')^2 - \mathrm{i}\hbar\frac{\partial}{\partial t}\right)_f G(\boldsymbol{r}'_f, t_f; \boldsymbol{r}'_i, t_i) = -\mathrm{i}\hbar\delta^3(\boldsymbol{r}'_f - \boldsymbol{r}'_i)\delta(t_f - t_i). \quad (32)$$

Here, $(\cdot)_f$ means that the operator acts upon $\boldsymbol{r}'_f$. The propagator, in addition, must fulfill the following conditions:

$$G(\boldsymbol{r}'_f, t_f; \boldsymbol{r}'_i, t_i) = 0, \qquad \text{if } t_f < t_i, \quad (33)$$

and

$$\lim_{t_f\to t_i} G(\boldsymbol{r}'_f, t_f; \boldsymbol{r}'_i, t_i) \to \delta^3(\boldsymbol{r}'_f - \boldsymbol{r}'_i). \quad (34)$$

To determine $G(\boldsymbol{r}'_f, t_f; \boldsymbol{r}'_i, t_i)$, and thus determine the temporal evolution of the initial wave function, we use the path-integral approach [32]. For $t_f > t_i$, we must have

$$G(\boldsymbol{r}'_f, t_f; \boldsymbol{r}'_i, t_i) = \int \mathcal{D}\boldsymbol{r}\, \exp\left(\frac{\mathrm{i}}{\hbar}\int_{t_i}^{t_f} \mathcal{L}(\boldsymbol{r}', \dot{\boldsymbol{r}}', t)\,\mathrm{d}t\right), \quad (35)$$

where $\mathcal{D}\boldsymbol{r}$ is an adequately defined functional measure. Taking advantage of the fact that the Lagrangian of this problem can be expressed as a quadratic form [33], the propagator becomes

$$G(\boldsymbol{r}'_f, t_f; \boldsymbol{r}'_i, t_i) = F(t_f, t_i)\, \exp\left(\frac{\mathrm{i}}{\hbar}\mathcal{S}_{\mathrm{cl}}(\boldsymbol{r}'_f, t_f, \boldsymbol{r}'_i, t_i)\right). \quad (36)$$

Notice that the pre-factor $F(t_f, t_i)$ is independent of $\boldsymbol{r}'_i$ and $\boldsymbol{r}'_f$. It can be evaluated by using the van Vleck determinant

$$F(t_f, t_i) = \sqrt{\det\left(\frac{\mathrm{i}}{2\pi\hbar}\frac{\partial^2\mathcal{S}_{\mathrm{cl}}}{\partial\boldsymbol{r}_i\partial\boldsymbol{r}_f}\right)} \quad (37)$$

obtaining

$$F(t_f, t_i) = \left(\frac{m}{2\pi\mathrm{i}\hbar s_1(t_f, t_i)}\right)^{3/2}. \quad (38)$$

Replacing equations (21) and (38) in the equation (36), we obtain the general form of the propagator,

$$G(\boldsymbol{r}'_f, t_f; \boldsymbol{r}'_i, t_i) = \left(\frac{m}{2\pi\mathrm{i}\hbar s_1(t_f, t_i)}\right)^{3/2} \times \exp\left(\frac{\mathrm{i}}{\hbar}\mathcal{S}_{\mathrm{cl}}(\boldsymbol{r}'_f, t_f, \boldsymbol{r}'_i, t_i)\right). \quad (39)$$

Thus, to know the temporal evolution of the wave function from a given initial state, it is only necessary to perform the integral displayed in equation (31).

Now we will analyze some particular cases and the consequences due to the rotation. Firstly, we will show an important consequence in the time-evolved wave function when one imposed the condition $\boldsymbol{\alpha}' = \boldsymbol{0}$ and chose a particular instant $t_f = t_{\mathrm{FO}}$ such that $s_2(t_{\mathrm{FO}}, t_i) = 0$. In this case, the classical action is given by equation (27), and therefore the propagator adopts the form

$$G(\boldsymbol{r}'_f, t_{\mathrm{FO}}; \boldsymbol{r}'_i, t_i) = \left(\frac{m}{2\pi\mathrm{i}\hbar s_1(t_{\mathrm{FO}}, t_i)}\right)^{3/2} \times \exp\left(\mathrm{i}\frac{m}{2\hbar s_1(t_{\mathrm{FO}}, t_i)}\Big(\dot{s}_1(t_{\mathrm{FO}}, t_i)\boldsymbol{r}'^2_f - 2\boldsymbol{r}'_f\cdot\mathsf{R}\big(\theta(t_{\mathrm{FO}}, t_i)\big)\boldsymbol{r}'_i\Big)\right). \quad (40)$$

and equation (31) can be expressed by

$$\psi(\boldsymbol{r}'_f, t_{\mathrm{FO}}) = \left(\frac{m}{2\pi\mathrm{i}\hbar s_1(t_{\mathrm{FO}}, t_i)}\right)^{3/2} \times \exp\left(\mathrm{i}\frac{m\dot{s}_1(t_{\mathrm{FO}}, t_i)}{2\hbar s_1(t_{\mathrm{FO}}, t_i)}\boldsymbol{r}'^2_f\right)\int \exp(-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}'_i)\psi_0(\boldsymbol{r}'_i)\,\mathrm{d}\boldsymbol{r}'_i, \quad (41)$$

where $\boldsymbol{k} = (m/\hbar s_1(t_{\mathrm{FO}}, t_i))\mathsf{R}(-\theta(t_{\mathrm{FO}}, t_i))\boldsymbol{r}'_f$. Using the definition of Fourier transform,

$$\hat{\zeta}(\boldsymbol{k}) = \int \mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}\zeta(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}, \quad (42)$$

the time-evolved wave function can be written in the form

$$\psi(\boldsymbol{r}'_f, t_{\mathrm{FO}}) = \left(\frac{m}{2\pi\mathrm{i}\hbar s_1(t_{\mathrm{FO}}, t_i)}\right)^{3/2} \times \exp\Big(\frac{\mathrm{i}\hbar}{2m}\dot{s}_1(t_{\mathrm{FO}}, t_i)s_1(t_{\mathrm{FO}}, t_i)\boldsymbol{k}^2\Big)\hat{\psi}_0(\boldsymbol{k}). \quad (43)$$

Thus, for an instant t_{FO} such that $s_2(t_{\mathrm{FO}}, t_i) = 0$, the time-evolved wave function is the Fourier transform of the initial wave function times a dynamical phase. This result is in close analogy with electron optics, and constitutes the basis of the diffraction theory [34].

Finally, let us analyze the quantum interference phenomena due to rotation. From equation (39) the case of a quadratic potential ($\boldsymbol{\alpha}' = 0$) in a non-inertial frame can be written as

$$G(\boldsymbol{r}'_f, t_f; \boldsymbol{r}'_i, t_i) = \left(\frac{m}{2\pi\mathrm{i}\hbar s_1(t_f, t_i)}\right)^{3/2} \exp\left(\frac{\mathrm{i}m}{2\hbar s_1(t_f, t_i)} \times \left(\dot{s}_1(t_f, t_i)\boldsymbol{r}'^2_f + s_2(t_f, t_i)\boldsymbol{r}'^2_i - 2\boldsymbol{r}'_f\cdot\mathsf{R}(\theta(t_f, t_i))\boldsymbol{r}'_i\right)\right) \quad (44)$$

and the propagator in the inertial frame ($\boldsymbol{\omega} = \boldsymbol{0}$, $\boldsymbol{\alpha} = 0$) can be cast in the form

$$G(\boldsymbol{r}_f, t_f; \boldsymbol{r}_i, t_i) = \left(\frac{m}{2\pi\mathrm{i}\hbar s_1(t_f, t_i)}\right)^{3/2} \exp\left(\frac{\mathrm{i}m}{2\hbar s_1(t_f, t_i)} \times (\dot{s}_1(t_f, t_i)\boldsymbol{r}^2_f + s_2(t_f, t_i)\boldsymbol{r}^2_i - 2\boldsymbol{r}_f\cdot\boldsymbol{r}_i)\right). \quad (45)$$

Therefore, the equation (44) can be written in the form

$$G(\boldsymbol{r}'_f, t_f; \boldsymbol{r}'_i, t_i) = G(\boldsymbol{r}_f, t_f; \boldsymbol{r}_i, t_i) \times \exp\left(\frac{\mathrm{i}m}{2\hbar s_1(t_f, t_i)}\boldsymbol{r}'_f\cdot\Big(\mathsf{I} - \mathsf{R}\big(\theta(t_f, t_i)\big)\Big)\boldsymbol{r}'_i\right). \quad (46)$$

From equation (46) an important consequence emerges: in quantum dynamics the rotation produces interference. The requirement for this interference phenomenon is that the phase factor be different to zero, that is $\theta(t_f, t_i) \neq 2n\pi$. Notice that from equation (46), if $\beta(t) = 0$, we obtain

$$G(\boldsymbol{r}'_f, t_f; \boldsymbol{r}'_i, t_i) = G(\boldsymbol{r}_f, t_f; \boldsymbol{r}_i, t_i) \times \exp\left(\frac{\mathrm{i}m}{2\hbar(t_f - t_i)} \boldsymbol{r}'_f \cdot \left(\mathsf{I} - \mathsf{R}\big(\theta(t_f, t_i)\big)\right) \boldsymbol{r}'_i\right). \qquad (47)$$

This last equation emphasizes the fact that the interference phenomenon is a consequence of a pure rotation.

4. Conclusion

We analyzed in detail the classical as well as the quantum behavior of a particle seen from a reference frame which rotates with respect to an inertial reference frame in a fixed axis, with a time-dependent angular velocity $\omega(t)$, and submitted to potentials which are linear and quadratic in position, with time-dependent coefficients. For the classical behavior, we find the analytical solution and use it to calculate the classical action explicitly, analyzing some particular cases. For example, we demonstrate that the difference in the classical action between two different reference frames could be zero if the integral of the modulus of the angular velocity is multiple of 2π.

The quantum behavior is obtained from the calculated classical action, via path-integral approach. In this way, we can exhibit a general, closed, quantum solution in terms of the time-dependent angular velocity, and both the time-dependent linear and quadratic potentials. From this solution one can easily analyze particular cases. For instance, when one of the solution of the Hill equation is zero, the time-evolved wavefunction can be expressed as Fourier transform of its initial wave function, in close analogy to the situation of electron optics, where it has a time-dependent magnetic field. Finally, we demonstrate that the propagator in the non-inertial frame can be written in terms of the propagator in the inertial reference system, stressing the fact that the quantum interference phenomenon is a direct consequence of the rotation. However, for certain specific time-functional forms of the angular velocity, and under certain particular times, it is possible to suppress this effect.

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References

[1] Yuen H P 1967 *Phys. Rev.* A **13** 2226
Agarwal G S and Kumar S A 1991 *Phys. Rev. Lett.* **67** 3665
[2] Paul W 1998 *Rev. Mod. Phys.* **62** 531
Brown L S 1991 *Phys. Rev. Lett.* **66** 527
[3] Cui S-M 1992 *Phys. Rev.* A **45** 5255
[4] Calvo M and Laroze D 2002 *Optik* **113** 429
[5] Calvo M 2004 *Ultramicroscopy* **99** 179
[6] Lewis Jr H R and Riesenfeld W B 1969 *J. Math. Phys.* **10** 1458
[7] Ji J-Y, Kim J K, Kim S P and Soh K-S 1995 *Phys. Rev.* A **52** 3352
[8] Pedrosa I A 1997 *Phys. Rev.* A **55** 3219
[9] Song D-Y 1999 *Phys. Rev.* A **59** 2616
[10] Yeon K H, Um C I and George Th F 2003 *Phys. Rev.* A **68** 052108
[11] Calvo M 1996 *Phys. Rev.* B **54** 5665
[12] Laroze D and Rivera R 2006 *Phys. Lett.* A **355** 348
[13] Fetter A L and Walecka J D 1980 One of the most remarkable effect of non inertial systems in classical dynamics is in Foucault Pendulum *Theoretical Mechanics of Particles and Continua* (New York: McGraw-Hill) pp 44–7
[14] Chandrasekhar S 1961 *Hydrodynamics and Hydromagnetic Stability* (Oxford: Oxford University Press)
[15] Osmanov Z N, Machabeli G Z and Rogava A D 2002 *Phys. Rev.* A **66** 042103
[16] Cabrera B, Gutfreund H and Little W A 1982 *Phys. Rev.* B **25** 6644
[17] Sakurai J J 1980 *Phys. Rev.* D **21** 2993
[18] Xu H-H and Tsai C-H 1990 *Phys. Rev.* A **41** 4046
[19] Shen J Q, Zhu H Y, Shi S L and Li J 2002 *Phys. Scr.* **65** 465
[20] Letaw J R and Pfautsch J D 1980 *Phys. Rev.* D **22** 1345
[21] Iyer B R 1982 *Phys. Rev.* D **26** 1900
[22] Hehl F W and Ni W-T 1990 *Phys. Rev.* D **42** 2045
[23] Page L A 1975 *Phys. Rev. Lett.* **35** 543
[24] Werner S A, Staudenmann J-L and Colella R 1979 *Phys. Rev. Lett.* **42** 1103
[25] Bonse U and Wroblewski Th 1983 *Phys. Rev. Lett.* **51** 1401
[26] Cui S-M and Xu H-H 1991 *Phys. Rev.* A **44** 3343
[27] Shen J Q and He S L 2003 *Phys. Rev.* B **68** 195421
[28] Gao X C, Xu J B and Qian T Z 1991 *Phys. Rev.* A **44** 7016
[29] José J V and Saletan E J 1998 *Classical Dynamics: A Contemporary Approach* (Cambridge: Cambridge University Press) pp 429–31
[30] Hill G W 1886 *Acta Math.* **8** 1
[31] Weinberger H F 1965 *A First Course in Partial Differential Equation* (New York: Blaiserll Publishing Company)
[32] Feynman R P and Hibbs H 1965 *Quantum Mechanics and Path Integrals* (New York: McGraw-Hill)
[33] Kleinert H 2004 *Path Integral in Quantum Mechanics, Statistics, Polymer Physics, and Financial Markets* 4th edn (Singapore: World Scientific)
Schulman L S 2006 *Techniques and Applications of Path Integration* (New York: Dover)
[34] Calvo M 2002 *Optik* **113** 233